Mines Paris Tech

Centre d'Automatique et Systèmes

ON OBSERVABILITY & OBSERVER FORMS

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Outline

1. Some observer problem formulations

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2. Some observability conditions

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3. Some observer forms

... and some advertising
cf G. Besançon, *Nonlinear observers and applications*, Springer 2007, and references therein...

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1. Some observer problem formulations
   - About motivations
   - About formalization
   - About methods
2. Some observability conditions
3. Some observer forms

State feedback, parameter identification, fault monitoring

$\Rightarrow$ *internal* information reconstruction from *I/O* data

$\Rightarrow$ observer pb

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How ? cf *observer forms*

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N.B. in general, $f(x, u, t) = f(x, u), h(x, u, t) = h(x)$
with $x \in X \subset \mathbb{R}^n, u \in U \subset \mathbb{R}^m, y \in Y \subset \mathbb{R}^p$ ($\equiv$ system Σ in the sequel)

Some observer problem formulations

About methods–overview

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For 'observer forms', typically :

- *global exponential tunable* observers ;
- $\dot{\hat{x}}(t) = f(\hat{x}(t), u(t)) + k(y(t) - h(\hat{x}(t)), t), \;\; k(0, t) = 0 \; \forall t \geq 0$

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  e.g. $k(y - h(\hat{(}x)), t) = k(t) \times [y(t) - h(\hat{x}(t))]$

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Some observability conditions

About 'structural' definition–Observability

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However "observability" is necessary for a 'tunable' observer.

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About 'structural' definition–Formal observability

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Observability [resp. at x_0] :
$\not\exists$ indistinguishable pair [resp. indistinguishable pair (x, x_0)].

N.B. Very general notion, even too general cf $\dot{x} = u$, $y = sin(x)$

$\Rightarrow$ a weaker notion of observability is more appropriate :

Local weak observability [resp. at x_0] :
$\forall x$ [resp. of x_0], $\exists$ a neighborhood U s.t. $\forall V \subset U$ neighborhood of x [resp. x_0], $\nexists$ indistinguishable state from x [resp. x_0] in V, as long as trajectories remain in V.

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In short : "distinguish every state from its neighbors without going too far"
$\Rightarrow$ more interesting in practice
$\Rightarrow$ with a 'simple' geometric characterization

Observation space :

The observation space $\mathcal{O}(h)$ for a system (Σ) is the smallest real vector space of $\mathcal{C}^\infty$ functions containing the components of h and closed under Lie derivation along $f_u := f(.,u)$ for any constant $u \in \mathbb{R}^m$
(i.e. $\forall \varphi \in \mathcal{O}(h), L_{f_u}\varphi \in \mathcal{O}(h)$, where $L_{f_u}\varphi(x) = \frac{\partial \varphi}{\partial x} f(x,u)$).

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Observability rank condition [resp. at x_0] :

$$\forall x, \quad dimd\mathcal{O}(h)\mid_x = n \quad \text{[resp. } dimd\mathcal{O}(h)\mid_{x_0} = n\text{]}$$

where $d\mathcal{O}(h)\mid_x := \{d\varphi(x), \varphi \in \mathcal{O}(h)\}$.

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Local weak observability characterization :
Observability rank condition (at x_0) $\Rightarrow$ local weak observability (at x_0).

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Ex. $\dot{x} = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} x, \; y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$ observable $\forall u \; cst \neq 0$, but not for $u = 0$

$\Rightarrow$ Need to look at the inputs.

Universal input [resp. on $[0, t]$] :
$u : \forall x_0 \neq x_0', \exists \tau \geq 0$ (resp. $\exists \tau \in [0, t]$) s.t. $h(\chi_u(\tau, x_0)) \neq h(\chi_u(\tau, x_0'))$.
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'Nice' case :
Uniformly observable systems (resp. locally) :
(Σ) is uniformly observable (UO) if every input is universal (resp. on $[0, t]$).

Ex. The system below is uniformly observable :

$$\begin{array}{rcl}\dot{x} & = & \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \\ & & & & 0 \\ \vdots & & & & 1 \\ 0 & \cdots & & & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \psi_n(x) \end{pmatrix} + \begin{pmatrix} \varphi_1(x_1) \\ \varphi_2(x_1, x_2) \\ \vdots \\ \varphi_{n-1}(x_1, \ldots, x_{n-1}) \\ \varphi_n(x_1, \ldots, x_n) \end{pmatrix} u \\ y & = & x_1; \quad x = (x_1 \ldots, x_n)^T \end{array}$$

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NB. u universal on $[0, t] \Leftrightarrow \int_0^t ||h(\chi_u(\tau, x_0)) - h(\chi_u(\tau, x_0'))||^2 d\tau > 0, \; x_0 \neq x_0'$

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NB.2 : uniform observability $\Rightarrow$ possible input-independent observer...

Some observability conditions

About 'excitation' conditions–Regularly persistent excitation

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Using universal inputs not enough : cf disturbance pb
Univ. inputs on $[t, t+T]$ (*persistent*) not enough either :cf 'vanishing info' pb

$\Rightarrow$ need of 'regular persistency' :
Regularly persistent inputs (RP) :
u is regularly persistent for (Σ) if :

$$\exists t_0, T : \forall x_{t-T}, x'_{t-T}, \ \forall t \geq t_0,$$
$$\int_{t-T}^{t} ||h(\chi_u(\tau, x_{t-T})) - h(\chi_u(\tau, x'_{t-T}))||^2 d\tau \geq \beta(\|x_{t-T} - x'_{t-T}\|)$$

for some class $\mathcal{K}$ function β.

Ex. For $\dot{x}(t) = A(u(t))x(t) + B(u(t))$, $y(t) = Cx(t)$, RP inputs are s.t.

$$\exists t_0, T, \alpha : \int_{t-T}^{t} \Phi_u^T(\tau, t-T) C^T C \Phi_u(\tau, t-T) d\tau \geq \alpha I > 0 \quad \forall t \geq t_0,$$

with $\Phi_u(\tau, t) : \dfrac{d\Phi_u(\tau, t)}{d\tau} = A(u(\tau))\Phi_u(\tau, t),\ \Phi_u(t, t) = I.$

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N.B. For $\dot{x}(t) = A(t)x(t)$, $y(t) = Cx(t)$: Kalman *Unif. Complete Obs.*

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N.B.2. Need of time T. For shorter times : 'short-time' excitation needed.

Locally regular inputs (LR) :

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$$\exists T_0, \alpha : \forall x_{t-T}, x'_{t-T},\ \forall T \leq T_0,\ \forall t \geq T,$$
$$\int_{t-T}^{t} ||h(\chi_u(\tau, x_{t-T})) - h(\chi_u(\tau, x'_{t-T}))||^2 d\tau \geq \beta(\|x_{t-T} - x'_{t-T}\|, \frac{1}{T})$$

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$$\int_{t-T}^{t} \Phi_u^T(\tau, t-T) C^T C \Phi_u(\tau, t-T) d\tau \geq \alpha \frac{1}{T} \begin{pmatrix} T & & & 0 \\ & T^2 & & \\ & & \ddots & \\ 0 & & & T^n \end{pmatrix}^2$$

N.B. LR inputs make observability $\simeq$ linear one.

Observer design needs :

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- If rank condition not satisfied, the system might be turned into :

$$\begin{aligned} \dot{\zeta}_1 &= f_1(\zeta_1, \zeta_2, u) \\ \dot{\zeta}_2 &= f_2(\zeta_2, u) \\ y &= h_2(\zeta_2) \end{aligned}$$

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  with f_2, h_2 rank observable.
- If system not observable, but s.t. :

$$\forall u : x_0,\ x_0' \text{ indistinguishable}, \chi_u(t, x_0) - \chi_u(t, x_0') \rightarrow 0$$

  an observer might still be designed (*detectability*).

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Summary-cont'ed

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  - For non-uniform observability, *non-uniform observers*.

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$\Rightarrow$ observer forms $\equiv$ uniform (cf Luenberger) & non-uniform (cf Kalman)

Outline

Basic (LTI) system :

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If (A, C) is observable, then $\exists$ an observer :

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Indeed : $e = \hat{x} - x \Rightarrow \dot{e} = (A - KC)e$

System with additive I/O nonlinearities :

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System with additive Lipschitz nonlinearities :

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(u(t), x(t)) \\ y(t) &= Cx(t) \end{aligned}$$

with B globally Lipschitz $/x$,unif.$/u$
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First idea :

If $\exists K$ and P, Q positive definite s.t.

$$\begin{aligned} P(A - KC) + (A - KC)^T P &= -Q \\ \frac{eigmin(Q)}{2eigmax(P)} &> \gamma \end{aligned}$$

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Pb : Find K, P, Q...

System with additive triangular nonlinearities :

$$\begin{array}{rcl} \dot{x}(t) & = & A_0 x(t) + B(u(t), x(t)) \\ y(t) & = & C_0 x(t) \end{array}$$

with $A_0 = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{pmatrix}, \quad C_0 = (1\ 0 \cdots\ 0),\ x \in \mathbb{R}^n,\ y \in \mathbb{R}$

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Result [High Gain Observer] :

If B globally Lipschitz $/x$,unif.$/u$: $\dfrac{\partial B_i}{\partial x_j}(u, x) = 0$ *for* $j \geq i + 1$,

$$\text{then } \exists \text{ obs. } \dot{\hat{x}} = A_0 \hat{x} + B(u, \hat{x}) - \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda^n \end{pmatrix} K_0 (C_0 \hat{x} - y)$$

with K_0 s.t. $A_0 - K_0 C_0$ stable, and λ large enough.

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- Possible *adaptive gain* implementation

$$\dot{\lambda}(t) = L(\lambda(t), \int_{t-T}^{t} \|h(\chi_u(\tau, z_{t-T})) - y(\tau)\|^2).$$

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- For observer form 'characteristic' of uniform observability

System :

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t), \; A(t), C(t) \text{ uniformly bounded.} \end{aligned}$$

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Result [Kalman] :

If $(A(t), C(t))$ is uniformly completely observable, then $\exists$ an observer :

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) - K(t)(C(t)\hat{x}(t) - y(t))$$

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N.B. K is to be chosen s.t.

$$\begin{array}{rcl} \dot{M}(t) & = & A(t)M(t)+M(t)A^T(t)-M(t)C^T(t)W^{-1}C(t)M(t)+V+\delta M(t) \\ M(0) & = & M_0 = M_0^T > 0,\ W = W^T > 0 \\ K(t) & = & M(t)C^T(t)W^{-1};\ \text{with } \delta > 2\|A(t)\|\ \forall t, \text{ or } V = V^T > 0\,; \end{array}$$

The rate of convergence can be arbitrarily chosen via δ, V.

System with additive *and multiplicative* I/O nonlinearities (*state affine systems*) :

$$\begin{aligned} \dot{x}(t) &= A(u(t), Cx(t))x(t) + B(u(t), Cx(t)) \\ y(t) &= Cx(t) \end{aligned}$$

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Result :

If u is regularly persistent for the system in the sense that it makes A, C uniformly completely observable, then $\exists$ observer :

$$\dot{\hat{x}}(t) = A(u(t), y(t))\hat{x}(t) + B(u(t), y(t)) - K(t)(C(t)\hat{x}(t) - y(t))$$

with $K(t)$ as in Kalman observer.

System with additive triangular nonlinearities
and multiplicative I/O nonlinearities :

$$\begin{aligned}\dot{x}(t) &= A_0(u(t), Cx(t))x(t) + B(u(t), x(t)) \\ y(t) &= C_0 x(t)\end{aligned}$$

with $A_0(u, Cx) = \begin{pmatrix} 0 & a_{12}(u, Cx) & & 0 \\ & & \ddots & \\ & & & a_{n-1n}(u, Cx) \\ 0 & & & 0 \end{pmatrix}$ bounded,

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$\rightarrow$ combine high gain and Kalman...

Result :

If B globally Lipschitz$/x$,unif.$/u$: $\frac{\partial B_i}{\partial x_j}(x,u) = 0 \; for \; j \geq i+1$ and u locally regular in the sense that it makes $v(t) := \begin{pmatrix} u(t) \\ C\chi_u(t,x_0) \end{pmatrix}$ locally regular for $\dot{x}(t) = A(v(t))x(t)$, $y(t) = Cx(t)$ for any x_0,
then $\exists$ an observer :

$$\dot{\hat{x}} = A_0(u,y)\hat{x} + \varphi(\hat{x},u) - \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda^n \end{pmatrix} K_0(t)(C_0\hat{x} - y)$$

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with $K_0(t)$ given by :

$$\begin{array}{rcl} \dot{M}(t) & = & \lambda[M(t)A^T(u(t),y(t)) + A(u(t),y(t))M(t) \\ & & -M(t)C^TW^{-1}CM(t) + \delta M(t)] \\ M(0) & = & M^T(0) > 0, \; W = W^T > 0 \\ K(t) & = & M(t)C^TW^{-1} \end{array}$$

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- For observer form 'characteristic' of uniform observability

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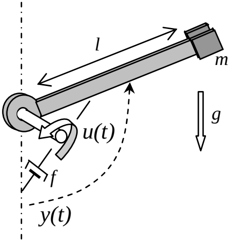

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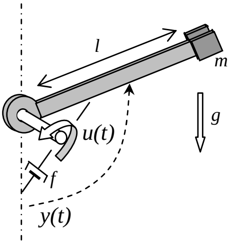

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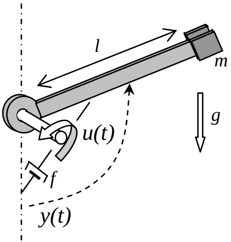
$$\Rightarrow \; ml^2\ddot{y}(t) + fl^2\dot{y}(t) + mglsin(y(t)) = u(t)$$



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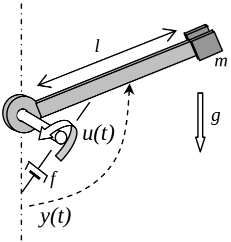

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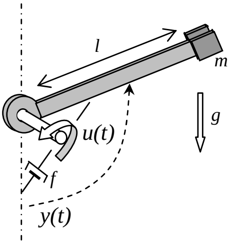

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$$\Rightarrow \begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a_1 sin(x_1) - a_2 x_2 + bu \\ y &= x_1 \end{cases}$$

$\Rightarrow$ Pb : x_1, x_2 ? $(\leftrightarrow x)$

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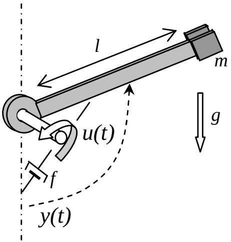

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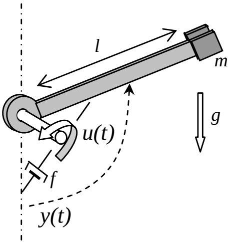

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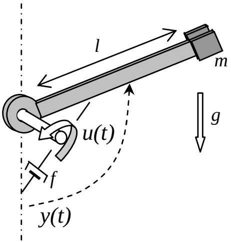

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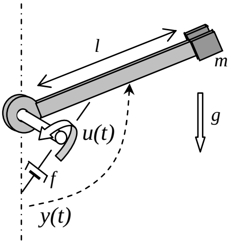

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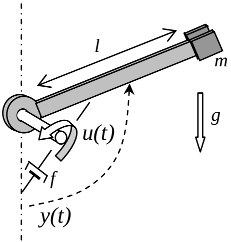

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$$\ddot{x}(t) - \gamma\left[1 - x^2(t)\right]\dot{x}(t) + \omega^2 x(t) = 0$$

with $y = x$ and $\dot{x}, \gamma, \omega$ to be estimated.

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Some observer forms

Example(s)...of observability condition

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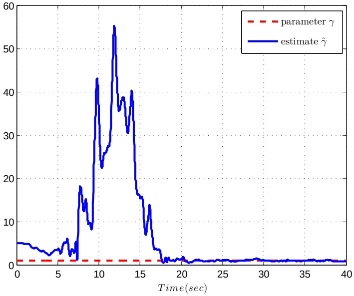

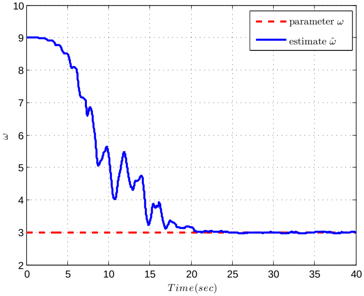

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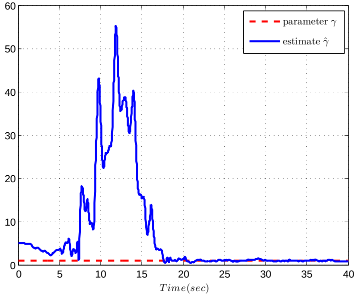

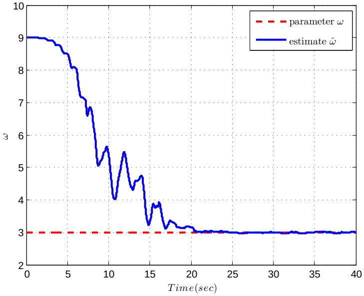

[cf *Automatica 2010*, to come]

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but also still some work to be done...