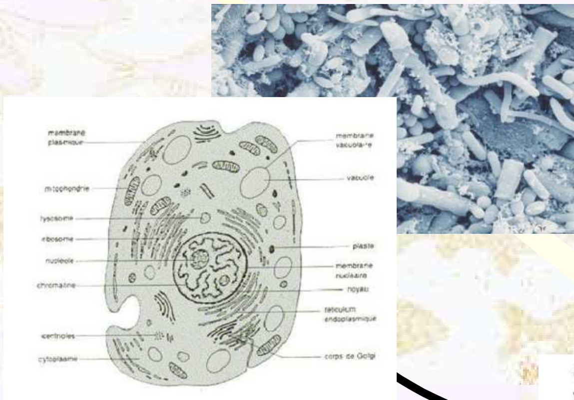
Développement d'observateurs par intervalles et applications aux systèmes biologiques

Olivier BERNARD

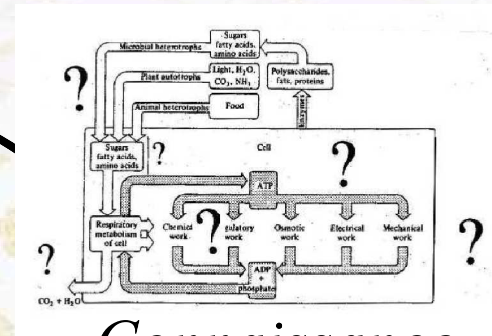
COMORE

INRIA-Sophia
CNRS-Villefranche

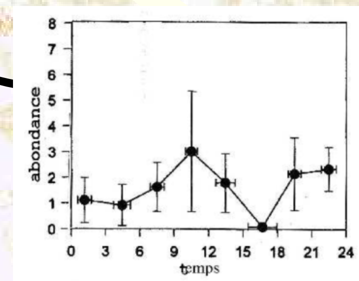
Introduction: les systèmes biologiques



Complexité



Connaissance imparfaite



Mesures incertaines

De plus en plus de procédés biotechnologiques qu'il faut surveiller !

Estimation d'état pour les systèmes biologiques

- **De nombreuses incertitudes**

$$\begin{cases} \dot{x} = f(x,u,p) + \epsilon(t) \\ y = h(x,u,p) + \eta(t) \end{cases}$$

Erreur de modèle ($\epsilon(t)$)

Erreur de sortie ($\eta(t)$)

- **Comment développer des estimateurs d'état ?**

  - **Approche stochastique**
  - **Estimation Robuste (ensembliste)**
  - **Estimation par intervalles**

Observateurs robustes

- Les observateurs supposent implicitement que le modèle est une bonne approximation du système réel...

  $\Rightarrow$ Difficile de construire un observateur exact qui garantirait

$$\|e(t)\| = \|\hat{x}(t) - x(t)\| \rightarrow 0 \text{ lorsque } \rightarrow \infty$$

- Nous présentons une approche qui consiste à borner les incertitudes

  $\Rightarrow$ Les bornes sur les variable estimées peuvent alors être déduites (Gouzé et al., 2000; Rapaport and Gouzé, 1999)

Observateurs robustes

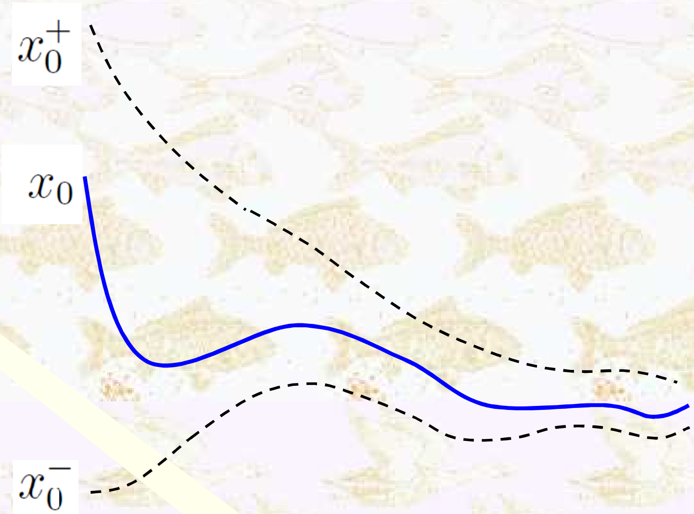
Definition 1 *A bounded error observer for system* (Σ) *is a dynamical system* $\dot{\hat{x}}(t) = \hat{f}(\hat{x}(t), \theta, u(t), y(t))$ *with*

$$\lim_{t\to\infty} ||\hat{x}(t) - x(t)|| \leq M$$

M *a positive real constant (depending on the system uncertainty on the system) such that* $M = 0$ *if the system is perfectly known.*

Concept

Bornes sur les termes incertains



$$x_0^- \leq x_0 \leq x_0^+ \rightarrow x^-(t) \leq x(t) \leq x^+(t)$$

Particulièrement adapté aux systèmes biotechnologiques

- Manque de capteurs hardware
- Incertitudes sur les modèles et les mesures disponibles

Observateurs par intervalles

- **Principe :**

$$(\mathcal{O}_\theta)\begin{cases} \dot{z}^- = f^-(z^-, z^+, y, u^-, u^+, p^-, p^+, \theta) \quad ; \quad z^-(t_0) = g^-(x_0^-, x_0^+) \\ \dot{z}^+ = f^+(z^-, z^+, y, u^-, u^+, p^-, p^+, \theta) \quad ; \quad z^+(t_0) = g^+(x_0^-, x_0^+) \\ x^- = h^-(z^-, z^+, y, u^-, u^+, p^-, p^+) \\ x^+ = h^+(z^-, z^+, y, u^-, u^+, p^-, p^+) \end{cases}$$

Ce système est un encadreur du système (Σ) si

$$x^-(0) \le x(0) \le x^+(0) \Rightarrow x^-(t) \le x(t) \le x^+(t) \quad \forall t \ge 0$$

Ce système est un observateur par intervalles de (Σ) s'il existe un réel positif *M*:

$$\lim_{t\to\infty} ||x^+(t) - x^-(t)|| \le M$$

Gouze et al. 2000

Principe: systèmes coopératifs

- **Définition :** *un système en dimension n est* ***coopératif*** *si sa matrice Jacobienne est positive en dehors de la diagonale sur un domaine convexe.*

$$\frac{Df}{Dx} = \begin{pmatrix} * & + & + \\ + & * & + \\ + & + & * \end{pmatrix}$$

Observateurs par intervalles

- **Principe: systèmes coopératifs:**

$$\frac{dx_1}{dt} = f(x_1, t) \quad ; \quad x_1(0) = x_{10}$$

$$\frac{dx_2}{dt} = g(x_2, t) \quad ; \quad x_2(0) = x_{20}$$

Propriété 1 *Si*

- $\forall z \in U,\ \forall t \geq 0,\ f(z,t) \leq g(z,t)$
- g *est coopérative*
- $x_{10} \leq x_{20}$

alors $x_1(t) \leq x_2(t)$ *pour* $t > 0$

Les inégalités doivent être considérées pour chaque composante

Observateurs par intervalles

- **Application :**

$$\dot{x}(t) = f(x(t), u(t))$$

Si *f* est coopérative, alors le système différentiel suivant encadre les solutions du système initial:

$$\dot{z}^+(t) = f(z^+(t), u(t)) \ ,$$
$$\dot{z}^-(t) = f(z^-(t), u(t)) \ ,$$
$$x^+(t) = z^+(t) \ ,$$
$$x^-(t) = z^-(t) \ ,$$

Si $x_0^- \leq x(0) \leq x_0^+$, alors $x^-(t) \leq x(t) \leq x^+(t)$ for all $t \geq 0$.

Si *f* est GAS, c'est un observateur par intervalle

$$(\Sigma): \quad \begin{cases} \dot{x}(t) = f(x(t), u(t), \theta) \\ y(t) = h(x(t), u(t), \theta) \end{cases} \qquad x(t_0) = x_0$$

Theorem 1 (Müller, 1926) *Assume that the application f in (Σ) is $\mathcal{C}^1$ in the domain $\Omega \times \mathcal{U} \times \Theta$, and that its Jacobian matrix is of fixed off diagonal signs.*

If (Σ) is initialized at t_0 such that $\mathrm{x}^-(t_0) \leq \mathrm{x}(t_0) \leq \mathrm{x}^+(t_0)$, then for any time $t \geq t_0$, we have:
$\mathrm{x}^-(t) \leq \mathrm{x}(t) \leq \mathrm{x}^+(t)$, *where x^- and x^+ are the solutions of the following differential systems:*

$$\dot{\mathrm{x}}^-(t) = f^-\left(\mathrm{x}^-(t), \mathrm{x}^+(t), \mathrm{u}(t), \theta\right) \tag{4}$$
$$\dot{\mathrm{x}}^+(t) = f^+\left(\mathrm{x}^-(t), \mathrm{x}^+(t), \mathrm{u}(t), \theta\right) \tag{5}$$

where the components of $f^-$and $f^+$are computed as follows from f, for $i = 1, \ldots, n_x$:

$$f_i^- = f_i\left(\delta_{f_i}^-\left(\mathrm{x}^-(t), \mathrm{x}^+(t)\right), \mathrm{u}(t), \theta\right) \tag{6}$$
$$f_i^+ = f_i\left(\delta_{f_i}^+\left(\mathrm{x}^-(t), \mathrm{x}^+(t)\right), \mathrm{u}(t), \theta\right) \tag{7}$$

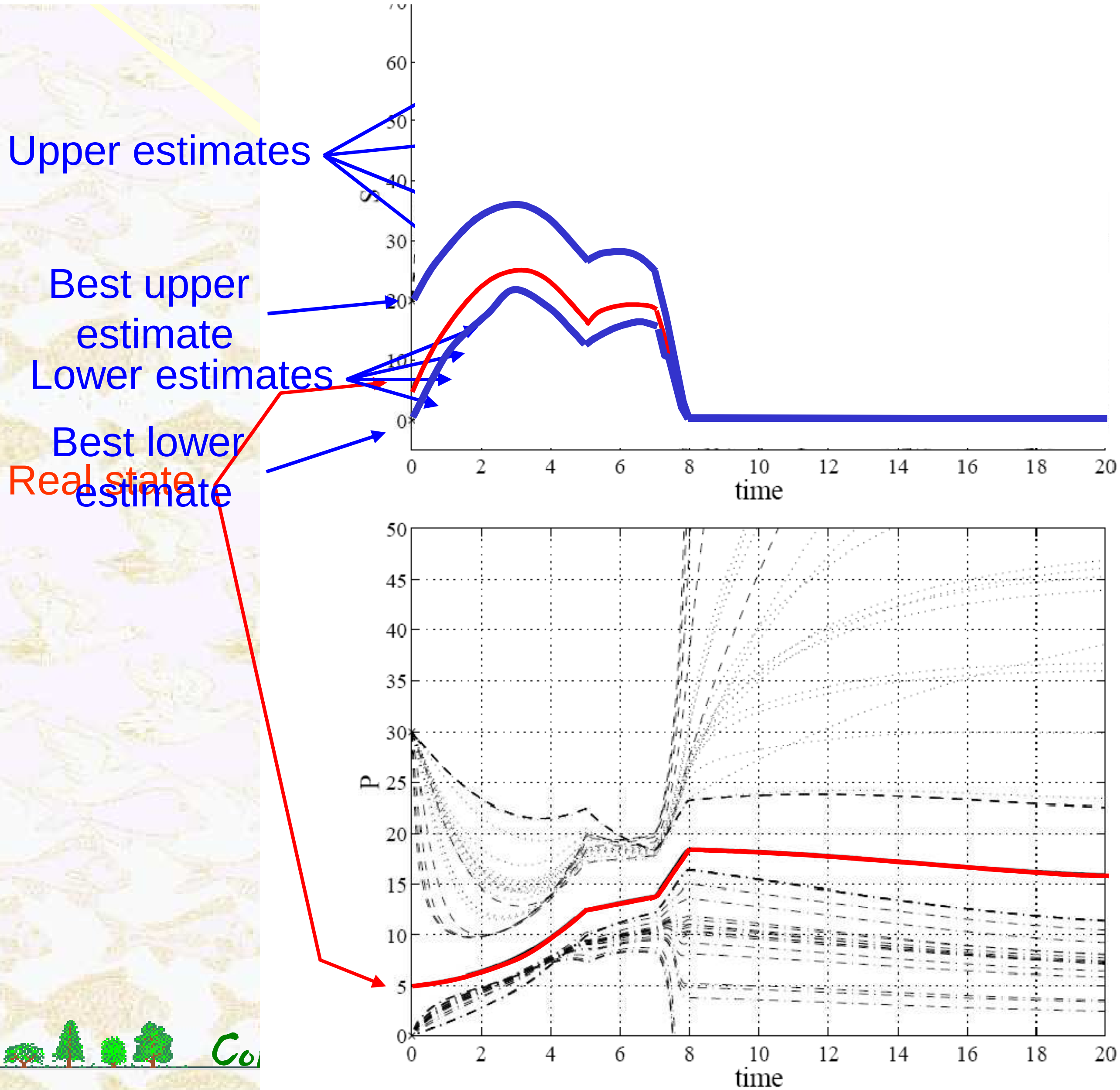
Sketch of the proof : *The following mapping is cooperative:*

$$\begin{pmatrix} \mathrm{x}^1 \\ \mathrm{x}^2 \end{pmatrix} \longrightarrow \begin{pmatrix} f^-(\mathrm{x}^1, -\mathrm{x}^2, \mathrm{u}, \theta) \\ -f^+(\mathrm{x}^1, -\mathrm{x}^2, \mathrm{u}, \theta) \end{pmatrix}$$

Indeed, the related Jacobian matrix J_x writes as follows:

$$J_x = \begin{bmatrix} \frac{Df^-}{D\mathrm{x}^-}(\mathrm{x}^1, -\mathrm{x}^2) & -\frac{Df^-}{D\mathrm{x}^+}(\mathrm{x}^1, -\mathrm{x}^2) \\ -\frac{Df^+}{D\mathrm{x}^-}(\mathrm{x}^1, -\mathrm{x}^2) & \frac{Df^+}{D\mathrm{x}^+}(\mathrm{x}^1, -\mathrm{x}^2) \end{bmatrix}$$

Possibilité de comparer les observateurs par intervalles entre eux… et de choisir le meilleur !



Bernard & Gouze, 2004

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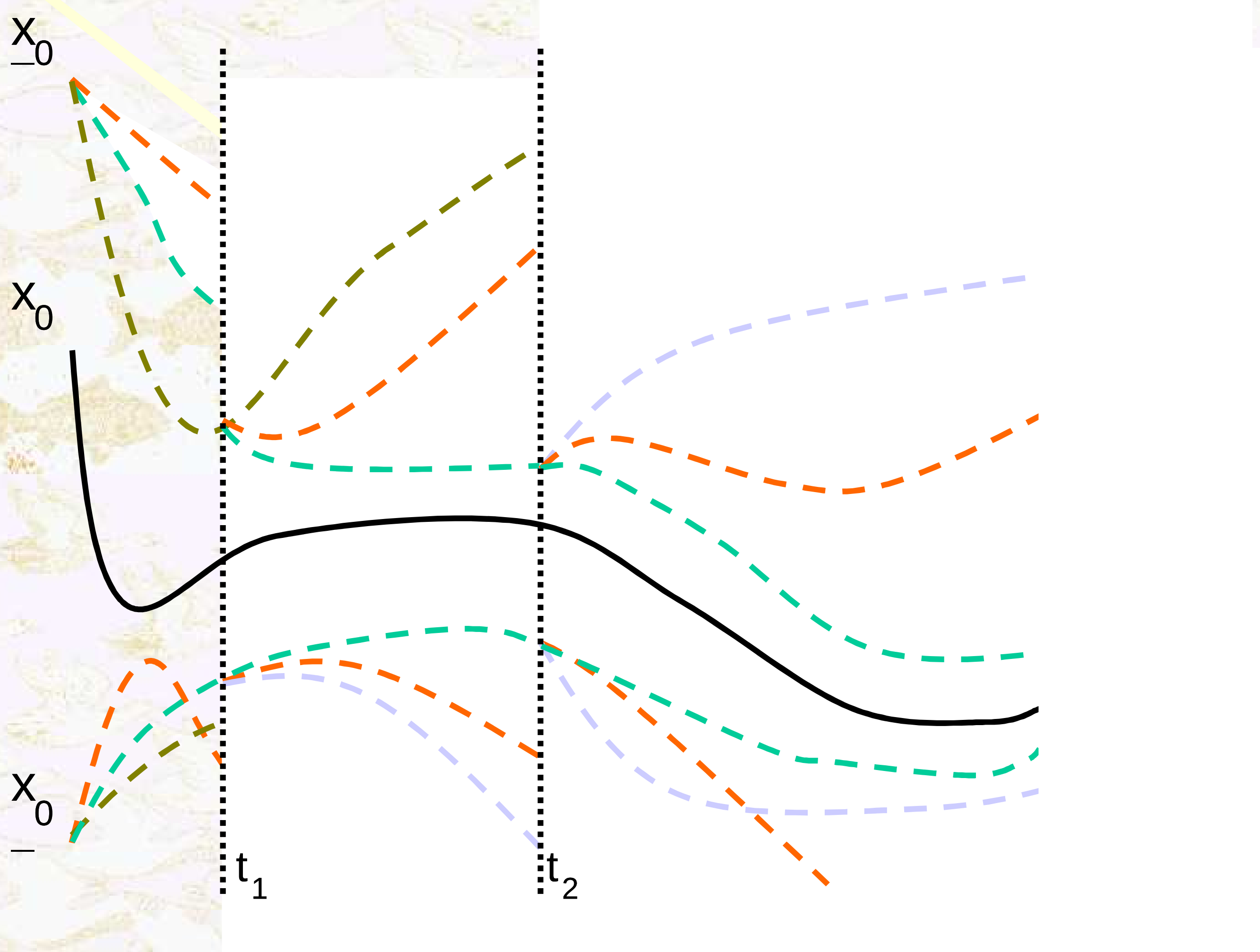
Reinitialisation avec la meilleure estimation



Bernard & Gouze, 2004

Certains encadreurs peuvent être instables !

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Application à des cultures de microalgues

- Modèle de Droop

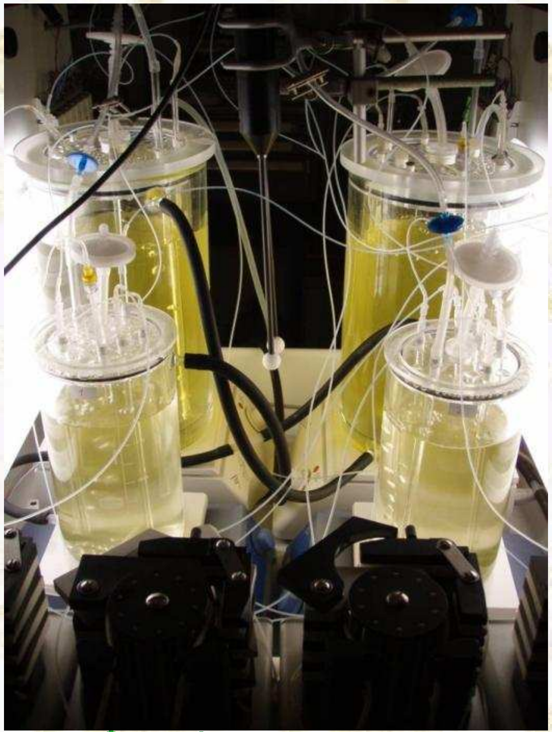
$$\dot{X} = -DX + \mu(Q)X$$
$$\dot{Q} = \rho(S) - \mu(Q)Q$$
$$\dot{S} = D\left(S_{in} - S\right) - \rho(S)X$$

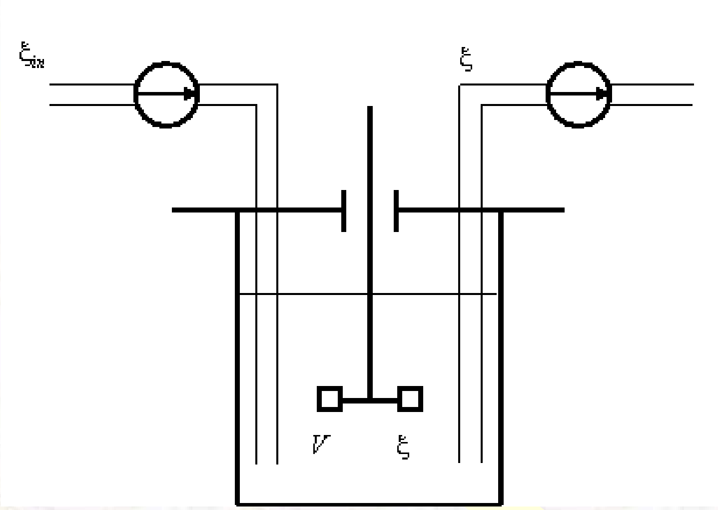
$$\mu(Q) = \bar{\mu}\left(1 - \frac{k_Q}{Q}\right)$$
$$\rho(S) = \rho_m \frac{S}{S + k_S}$$

X : biovolume $\left(\mu m^3 / L\right)$

Q : quota interne $\left(\mu mol / \mu m^3\right)$

S : substrat $\left(\mu mol / L\right)$





On mesure la biomasse X

Observateur direct

- Prédicteur

$$\dot{Q}_P^+ = \rho_m^+ \frac{S_P^+}{S_P^+ + k_S^-} - \bar{\mu}^- \left(1 - \frac{k_Q^+}{Q_P^+}\right) Q_P^+ \qquad \dot{Q}_P^- = \rho_m^- \frac{S_P^-}{S_P^- + k_S^+} - \bar{\mu}^+ \left(1 - \frac{k_Q^-}{Q_P^-}\right) Q_P^-$$

$$\dot{S}_P^+ = -\rho_m^- \frac{X_P^- S_P^+}{S_P^+ + k_S^+} + D^+ \left(S_{in}^+ - S_P^+\right) \qquad \dot{S}_P^- = -\rho_m^+ \frac{X_P^+ S_P^-}{S_P^- + k_S^-} + D^- \left(S_{in}^- - S_P^-\right)$$

$$\dot{X}_P^+ = \bar{\mu}^+ \left(1 - \frac{k_Q^-}{Q_P^+}\right) X_P^+ - D^- X_P^+ \qquad \dot{X}_P^- = \bar{\mu}^- \left(1 - \frac{k_Q^+}{Q_P^-}\right) X_P^- - D^+ X_P^-$$

Observateur direct

$$\dot{Q}^{+} = \rho_m^{+} \frac{S^{+}}{S^{+} + k_S^{-}} - \overline{\mu}^{-} \left(1 - \frac{k_Q^{+}}{Q^{+}}\right) Q^{+} \qquad \dot{S}^{+} = -\rho_m^{-} \frac{X^{-} S^{+}}{S^{+} + k_S^{+}} + D^{+}\left(S_{in}^{+} - S^{+}\right)$$

$$\dot{Q}^{-} = \rho_m^{-} \frac{S^{-}}{S^{-} + k_S^{+}} - \overline{\mu}^{+} \left(1 - \frac{k_Q^{-}}{Q^{-}}\right) Q^{-} \qquad \dot{S}^{-} = -\rho_m^{+} \frac{X^{+} S^{-}}{S^{-} + k_S^{-}} + D^{-}\left(S_{in}^{-} - S^{-}\right)$$

$$X^{+} = \begin{cases} \min\left(X_P^{+}, y_X^{+}\right) & \text{à chaque instant d'échantillonnage} \\ X_P^{+} & \text{sinon} \end{cases}$$

$$X^{-} = \begin{cases} \max\left(X_P^{-}, y_X^{-}\right) & \text{à chaque instant d'échantillonnage} \\ X_P^{-} & \text{sinon} \end{cases}$$

Observateur après transformation

- Transformation d'état permettant d'éliminer les termes cinétiques

$$Z = XQ \qquad \alpha = Z + S$$

$$\dot{Z} = -DZ + \rho_m \frac{XS}{S + k_S} \qquad \dot{\alpha} = -D\alpha + DS_{in}$$

- Observateurs par intervalles :

$$\dot{Z}^+ = -D^- Z^+ + \rho_m^+ \frac{X^+ S^+}{S^+ + k_S^-} \qquad \dot{Z}^- = -D^+ Z^- + \rho_m^- \frac{X^- S^-}{S^- + k_S^+}$$

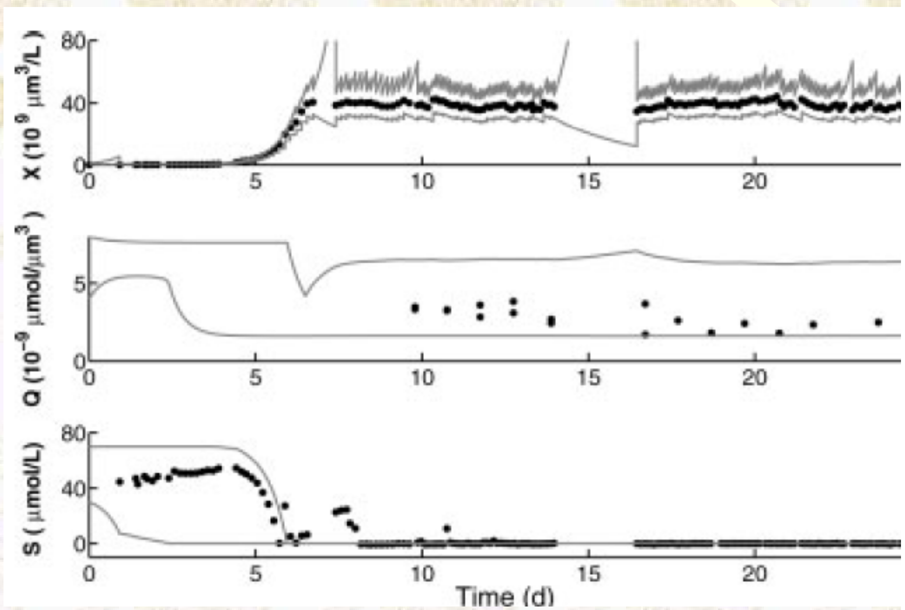
$$\dot{\alpha}^+ = D^+ \left(S_{in}^+ - \alpha^+ \right) \qquad \dot{\alpha}^- = D^- \left(S_{in}^- - \alpha^- \right)$$

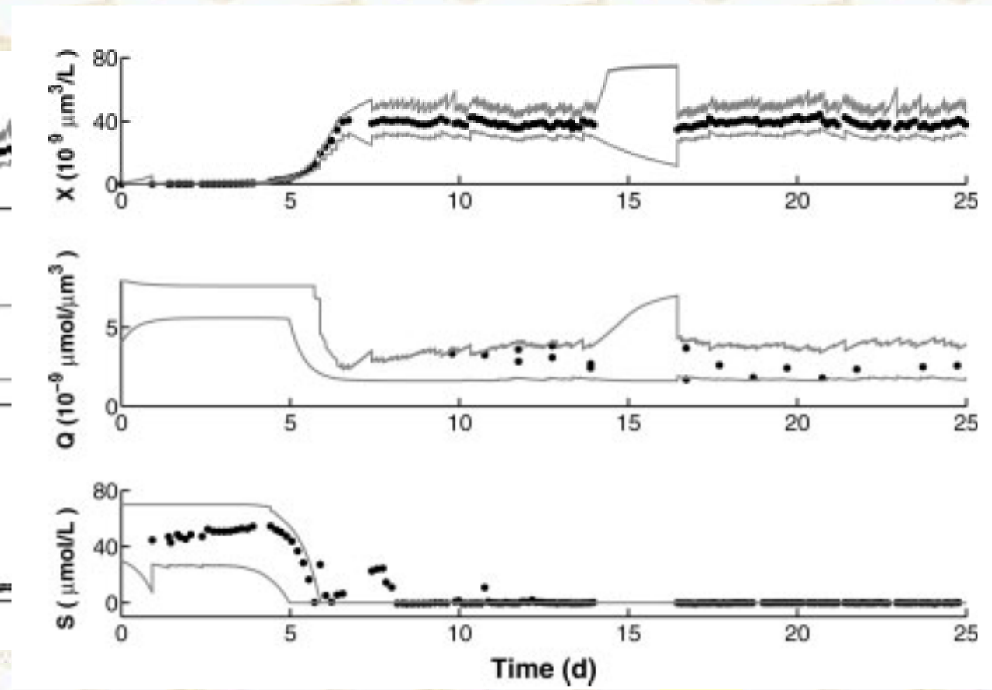
$$S^+ = \alpha^+ - Z^- \qquad Q^+ = Z^+ / X^-$$

$$S^- = \alpha^- - Z^+ \qquad Q^- = Z^- / X^+$$

Application expérimentale

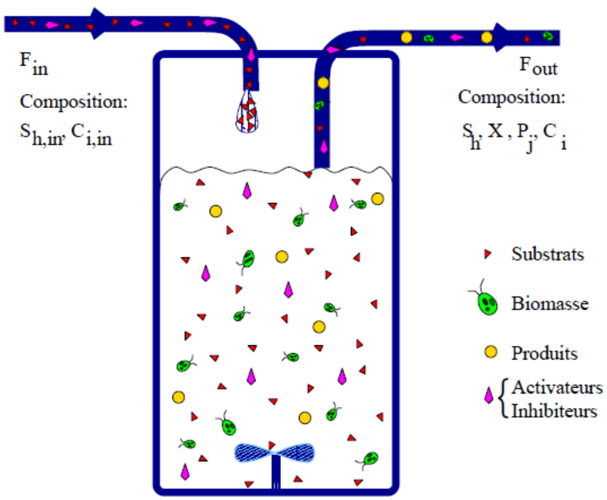
- **Observateur direct**
- **Observateurs couplés**





G. Goffaux, A. Vande Wouwer, O. Bernard. "Biotech. Progress", 2009

A simple hybrid observer



$$\begin{cases} \dot{x} = \mu(.)x - \alpha d x \\ \dot{s} = -k\mu(.)x + d s_{in} - d s \\ y = s \end{cases}$$

$\mu(.) = \mu(s)$: growing rate function

$\mu(0) = 0$: no growth if there is no substrate

$\mu(s)$ is poorly known

The objective is to adapt the observer design to the available knowledge of the growth rate, for the estimation of the biomass x

Asymptotic observer

Let us define $z = x + s/k$. Then the dynamics of z is :

$$\dot{z} = \frac{ds_{in}}{k} - \alpha d z - d\frac{1-\alpha}{k}s$$

An asymptotic observer is given by

$$\dot{\hat{z}} = \frac{ds_{in}}{k} - \alpha d \hat{z} - d\frac{1-\alpha}{k}s$$

It converges towards z with a rate of convergence αd.

Then, $\hat{x}$ can be estimated as follows: $\hat{x} = \hat{z} - \frac{s}{k}$.

✓ It does not require any a-priori knowledge for the growth rate
× It cannot be tuned (fixed convergence rate)

Consider the change of variable $(s, x) \rightarrow (s, z)$ with:

$$z = x + \frac{\theta}{k} s$$

The dynamics of z is :

$$\dot{z} \;=\; (1-\theta)\mu(s)(z - \tfrac{\theta}{k}s) - \alpha d z + \tfrac{\theta}{k} d s_{in} - \tfrac{\theta(1-\alpha)}{k} d s$$

Then a bounded error observer is (Lemesle and Gouzé, 2005):

$$\left\{ \begin{array}{rcl} \dot{\hat{z}} & = & (1-\theta)\hat{\mu}(s)(\hat{z} - \tfrac{\theta}{k}s) - \alpha d\hat{z} + \tfrac{\theta}{k} d s_{in} - \tfrac{\theta(1-\alpha)}{k} d s \\ \hat{x} & = & \hat{z} - \tfrac{\theta}{k} s \end{array} \right. \quad (1)$$

✓ High gain θ: high convergence rate but poor accuracy.
✓ $\theta = 1$: Low convergence rate but high accuracy.

Asymptotic interval observer

Initial condition hypothesis: $x^-(0) \leq x(0) \leq x^+(0)$

$$\left\{ \begin{array}{rcl} \dot{z}^+ & = & d\frac{s_{in}(t)}{k} - \alpha d z^+ - \frac{1-\alpha}{k} d s(t) \\ \dot{z}^- & = & d\frac{s_{in}(t)}{k} - \alpha d z^- - \frac{1-\alpha}{k} d s(t) \\ \multicolumn{3}{l}{x^+ = z^+ - \frac{s(t)}{k}, \quad x^- = z^- - \frac{s(t)}{k}} \end{array} \right.$$

Proof: if we denote $e^+ = x^+ - x = z^+ - z$, it can be easily seen that $\dot{e}^+ \geq 0$ when $e^+ = 0$.

✓ We obtain a bound on the accuracy of the observer.
✓ The same principle applies for asymptotic observer in dimension n.

Hybrid observer

Hypothesis 1 (H') *We assume that the growth rate is bounded by two* **known** *functions:*

$$0 \leq \mu^-(s) \leq \mu(s) \leq \mu^+(s)$$

with $\mu^-(s)$ positive for s positive and $\mu^+(0) = 0$.

Depending on the values of θ the following systems are interval estimates of the variable $x(t)$:

- If $\theta \leq 0$

$$\left\{\begin{array}{lll} \dot{z}^+ &=& (1-\theta)\mu^+(s)z^+ - (1-\theta)\frac{\theta}{k}\mu^-(s)s \\ & & -\alpha d z^+ + \frac{\theta}{k} d s_{in} - \frac{\theta(1-\alpha)}{k} d s \\ \dot{z}^- &=& (1-\theta)\mu^-(s)z^- - (1-\theta)\frac{\theta}{k}\mu^+(s)s \\ & & -\alpha d z^- + \frac{\theta}{k} d s_{in} - \frac{\theta(1-\alpha)}{k} d s \\ x^+ = z^+ - \frac{\theta s}{k}, x^- = z^- - \frac{\theta s}{k} \end{array}\right.$$

- If $0 \leq \theta \leq 1$

$$\left\{ \begin{array}{rcl} \dot{z}^+ &=& (1-\theta)\mu^+(s)z^+ - (1-\theta)\frac{\theta}{k}\mu^-(s)s \\ && -\alpha d z^+ + \frac{\theta}{k} d s_{in} - \frac{\theta(1-\alpha)}{k} d s \\ \dot{z}^- &=& (1-\theta)\mu^-(s)z^- - (1-\theta)\frac{\theta}{k}\mu^+(s)s \\ && -\alpha d z^- + \frac{\theta}{k} d s_{in} - \frac{\theta(1-\alpha)}{k} d s \\ \multicolumn{3}{l}{x^+ = z^+ - \frac{\theta s}{k}, x^- = z^- - \frac{\theta s}{k}} \end{array} \right.$$

- If $\theta \geq 1$

$$\left\{ \begin{array}{rcl} \dot{z}^+ &=& (1-\theta)\mu^-(s)z^+ - (1-\theta)\frac{\theta}{k}\mu^+(s)s \\ && -\alpha d z^+ + \frac{\theta}{k} d s_{in} - \frac{\theta(1-\alpha)}{k} d s \\ \dot{z}^- &=& (1-\theta)\mu^+(s)z^- - (1-\theta)\frac{\theta}{k}\mu^-(s)s \\ && -\alpha d z^- + \frac{\theta}{k} d s_{in} - \frac{\theta(1-\alpha)}{k} d s \\ \multicolumn{3}{l}{x^+ = z^+ - \frac{\theta s}{k}, x^- = z^- - \frac{\theta s}{k}} \end{array} \right.$$

where $\mu^+(s) = \dfrac{\mu_h^+ s}{s + k_s + s^2/k_i}$ and $\mu^-(s) = \dfrac{\mu_h^- s}{s + k_s + s^2/k_i}$

Sketch of the proof

Proof: Let us demonstrate for the case where $\theta \geq 1$.
The dynamics for this error is:

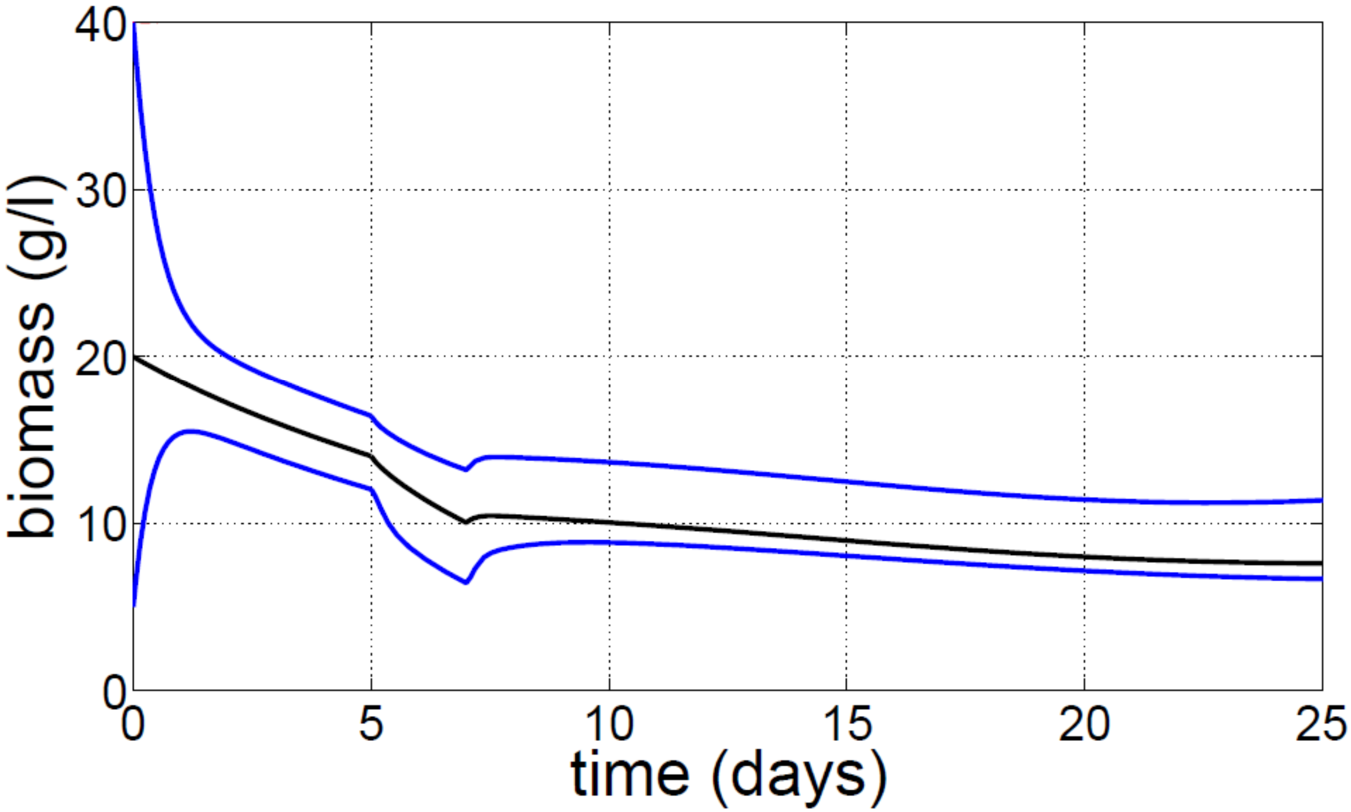
$$\dot{e}^+ \;=\; (1-\theta)(\mu^-(s)z^+ - \mu(s)z) - \alpha d e^+ - (1-\theta)\tfrac{\theta}{k}(\mu^+(s) - \mu(s))s$$

Now if we consider the first time t_0 such that $e^+(t_0) = 0$ (meaning $z^+(t_0) = z(t_0)$), we get:

$$\dot{e}^+(t_0) = (1-\theta)(\mu^-(s) - \mu(s))z - (1-\theta)\frac{\theta}{k}(\mu^+(s) - \mu(s))s \geq 0$$

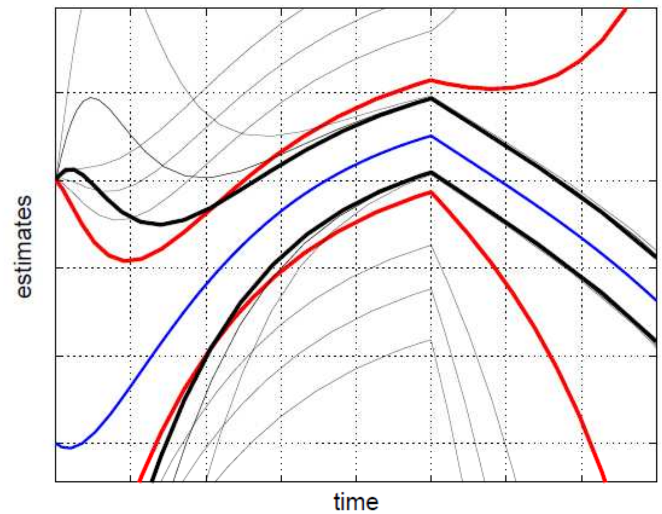
it proves therefore that $e^+(t_0)$ is increasing, and therefore that it will stay non negative. The same property holds for $e^- = x - x^- = z - z^-$.

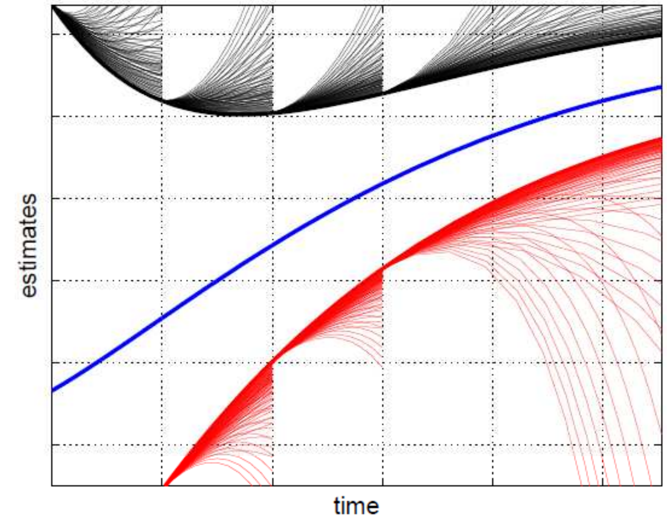
$s_{in} = 55 + 25cos(t/5).$



Example of application of an interval observer.

Bundle of observers

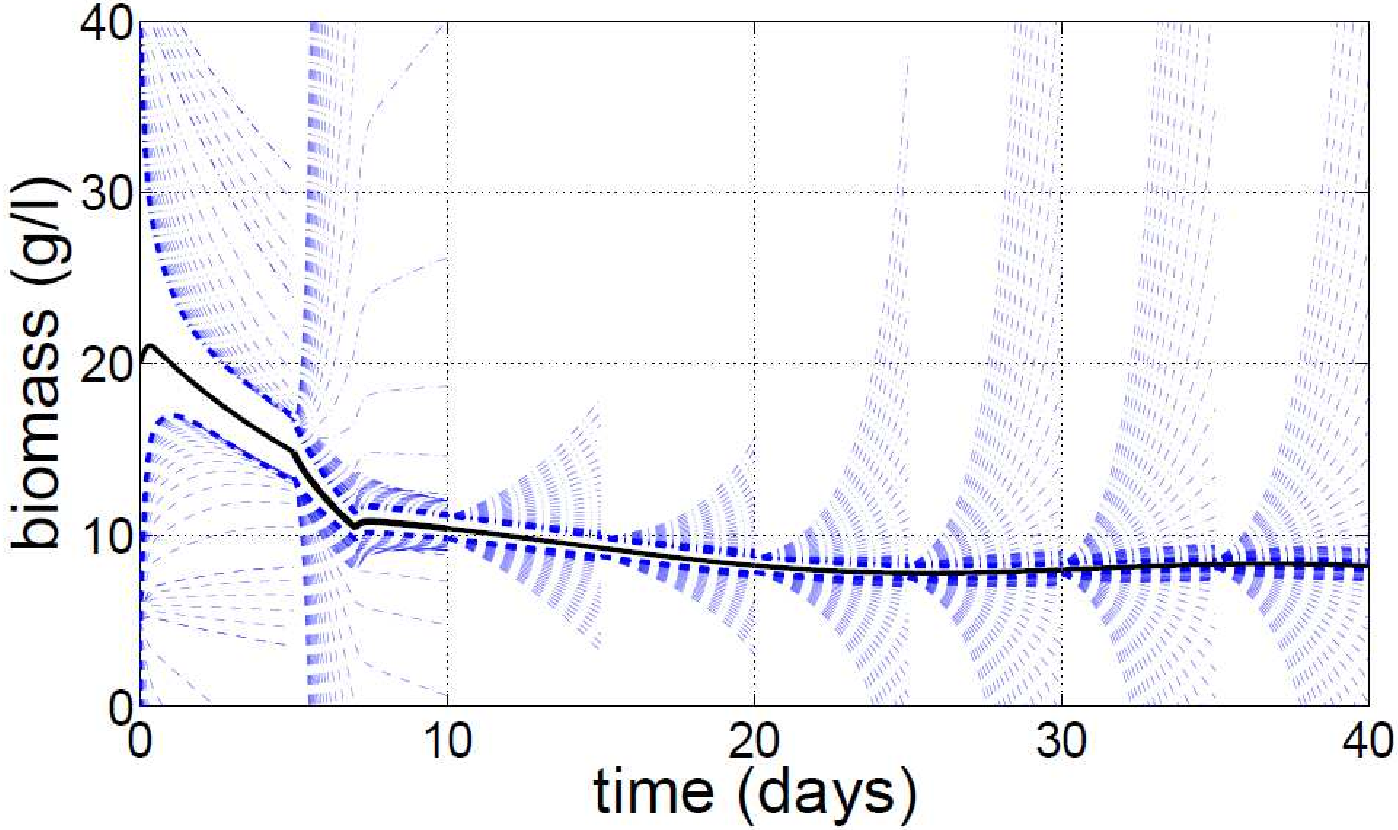




- Different observers are obtained for each value of θ
- $[X_0^-(t_r), X_0^+(t_r)] = [\max\{X_{\Theta_i}^-\}(t_r), \min\{X_{\Theta_i}^+\}(t_r)]$, where t_r is the reinitialisation time.
- Regular reinitialisation let us improve the convergence rate of the observer.

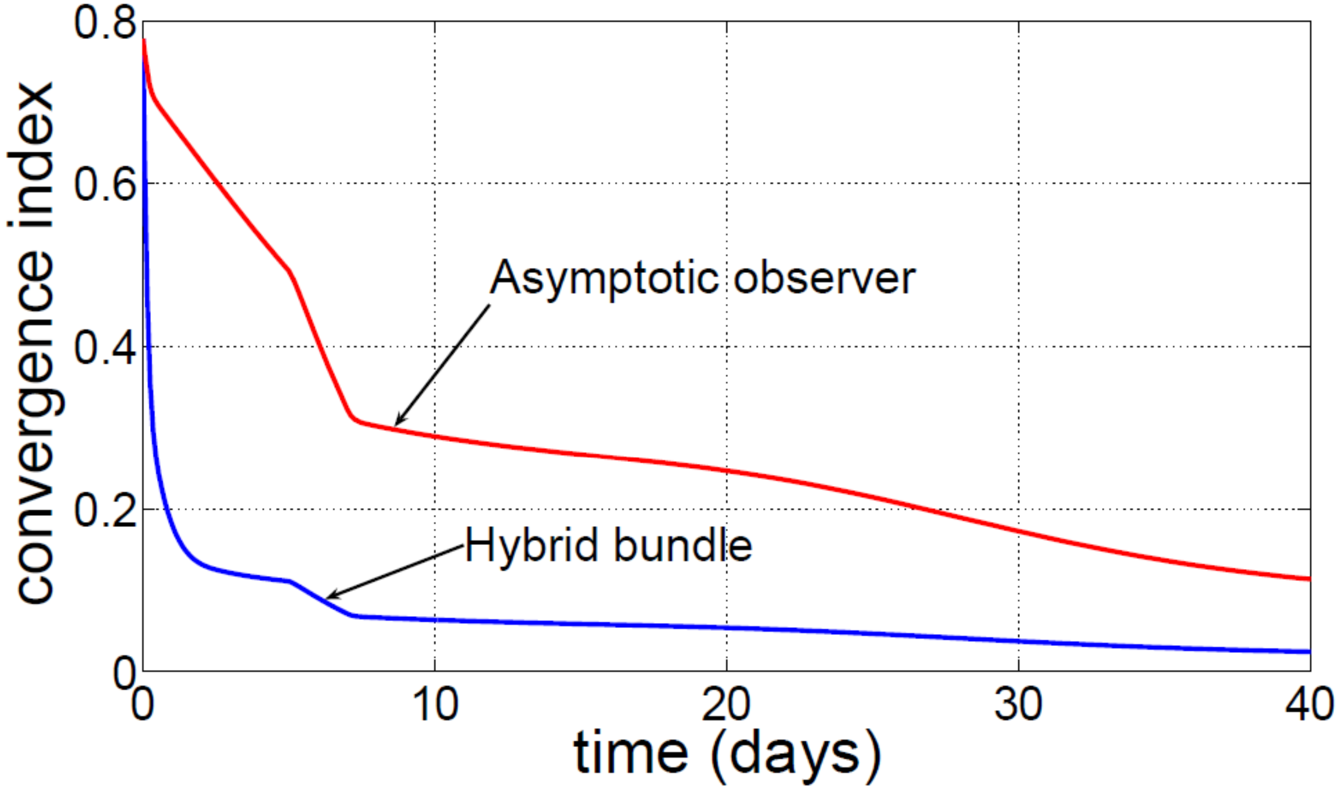
Improvement of the estimate by varying the observer gains

Gain θ takes 40 different values on the interval $[-20, 100]$.



Bundle of hybrid interval observers.

Gain θ takes 40 different values on the interval $[-20, 100]$.



Convergence index

$$cc = \frac{x^+ - x^-}{x^+ + x^-}$$

Including uncertainties

- s_{in} is supposed not to be known but bounded:

$$s_{in}^-(t) \leq s_{in}(t) \leq s_{in}^+(t)$$

- The measurements of s is supposed to be noisy: $y = s + \epsilon$, $|\epsilon| \leq m$. Three observers can be derived:

- If $\theta \leq 0$

$$\left\{ \begin{array}{lcl} \dot{z}^+ & = & (1-\theta)\bar{\mu}_m^+(y)z^+ - (1-\theta)\frac{\theta}{k}\bar{\mu}_m^+(y)(y+m) \\ & & -\alpha d z^+ + \frac{\theta}{k} d s_{in}^- - \frac{\theta(1-\alpha)}{k} d(y+m) \\ \dot{z}^- & = & (1-\theta)\bar{\mu}_m^-(y)z^- - (1-\theta)\frac{\theta}{k}\bar{\mu}_m^-(y)(y-m) \\ & & -\alpha d z^- + \frac{\theta}{k} d s_{in}^+ - \frac{\theta(1-\alpha)}{k} d(y-m) \\ x^+ = z^+ - \frac{\theta(y+m)}{k}, x^- = z^- - \frac{\theta(y-m)}{k} \end{array} \right.$$

- If $0 \leq \theta \leq 1$

$$\begin{cases} \dot{z}^+ = (1-\theta)\bar{\mu}_m^+(y)z^+ - (1-\theta)\frac{\theta}{k}\bar{\mu}_m^-(y)(y-m) \\ \qquad -\alpha d z^+ + \frac{\theta}{k} d s_{in}^+ - \frac{\theta(1-\alpha)}{k} d(y-m) \\ \dot{z}^- = (1-\theta)\bar{\mu}_m^-(y)z^- - (1-\theta)\frac{\theta}{k}\bar{\mu}_m^+(y)(y+m) \\ \qquad -\alpha d z^- + \frac{\theta}{k} d s_{in}^- - \frac{\theta(1-\alpha)}{k} d(y+m) \\ x^+ = z^+ - \frac{\theta(y-m)}{k}, x^- = z^- - \frac{\theta(y+m)}{k} \end{cases}$$

- If $\theta \geq 1$

$$\begin{cases} \dot{z}^+ = (1-\theta)\bar{\mu}_m^-(y)z^+ - (1-\theta)\frac{\theta}{k}\bar{\mu}_m^+(y)(y+m) \\ \qquad -\alpha d z^+ + \frac{\theta}{k} d s_{in}^+ - \frac{\theta(1-\alpha)}{k} d(y-m) \\ \dot{z}^- = (1-\theta)\bar{\mu}_m^+(y)z^- - (1-\theta)\frac{\theta}{k}\bar{\mu}_m^-(y)(y-m) \\ \qquad -\alpha d z^- + \frac{\theta}{k} d s_{in}^- - \frac{\theta(1-\alpha)}{k} d(y+m) \\ x^+ = z^+ - \frac{\theta(y-m)}{k}, x^- = z^- - \frac{\theta(y+m)}{k} \end{cases}$$

where $\bar{\mu}_m^+(y) = \max\{\mu^+(s)\}$ and $\bar{\mu}_m^-(y) = \min\{\mu^-(s)\}$
$s \in [y-m, y+m]$.

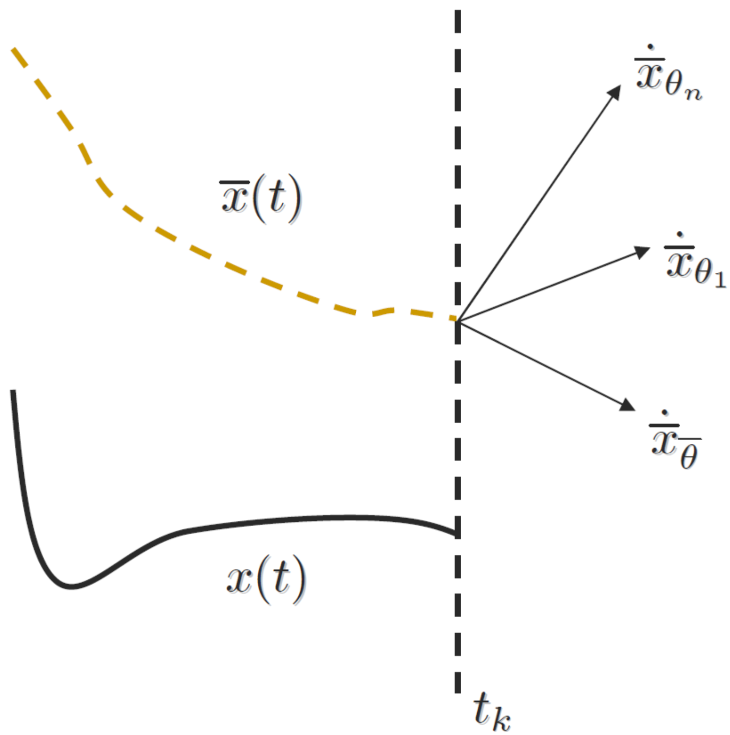
Optimality

Consider the functional

$$\overline{J}(\theta, x, \overline{x}, y) = \dot{\overline{x}}_\theta = (\dot{\overline{z}}_\theta - \theta(t)\dot{s} - \dot{\theta}(t)s)/k$$
$$\underline{J}(\theta, x, \underline{x}, y) = \dot{\underline{x}}_\theta = (\dot{\underline{z}}_\theta - \theta(t)\dot{s} - \dot{\theta}(t)s)/k$$

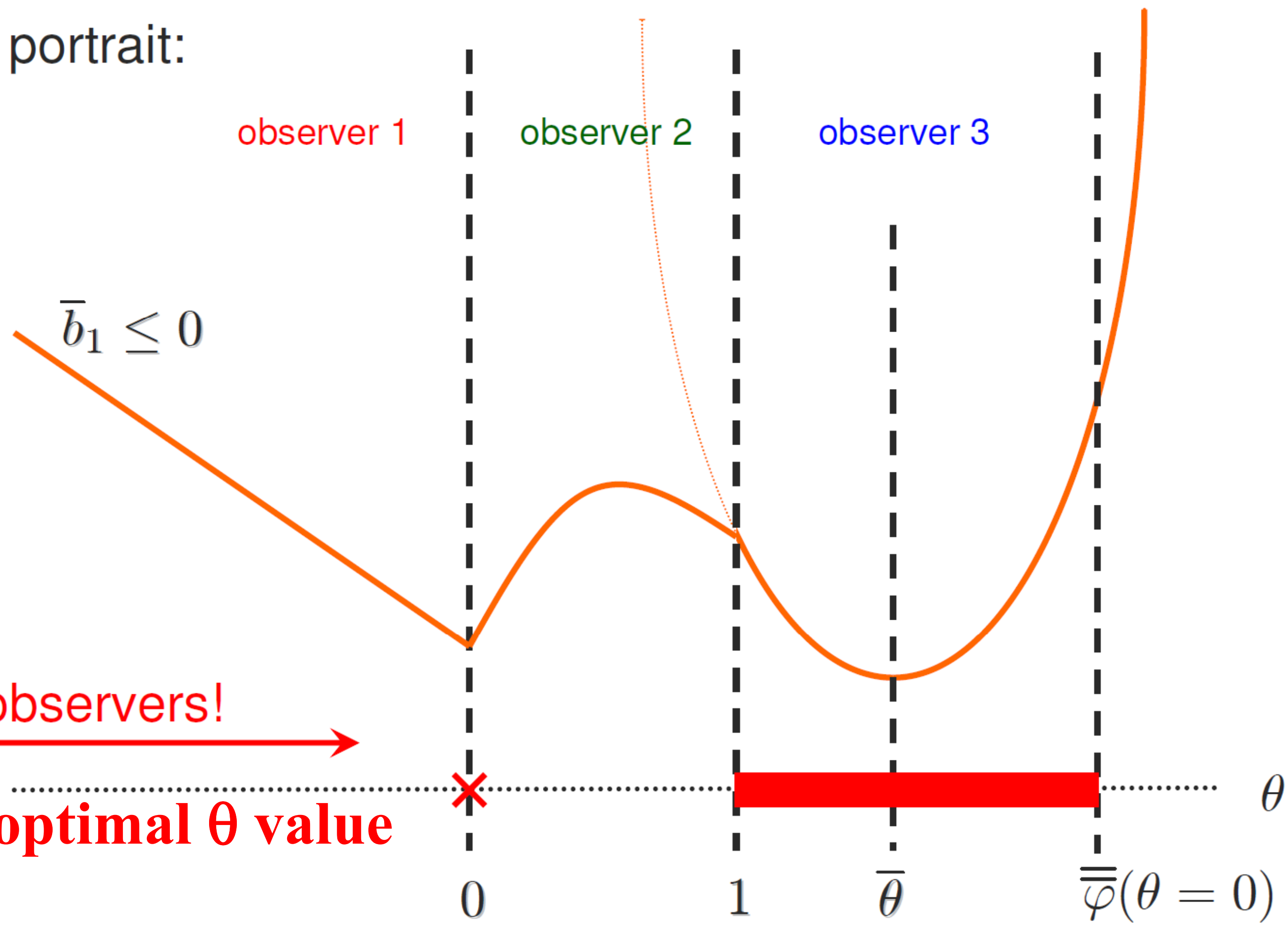
Look for $\underline{\theta}$ and $\overline{\theta}$ such that:

$$\overline{J}(\overline{\theta}(s, x, \overline{x})) = \min_\theta\{\overline{J}(\theta, s, x, \overline{x})\}$$
$$\underline{J}(\underline{\theta}(s, x, \underline{x})) = \max_\theta\{\underline{J}(\theta, s, x, \underline{x})\}$$



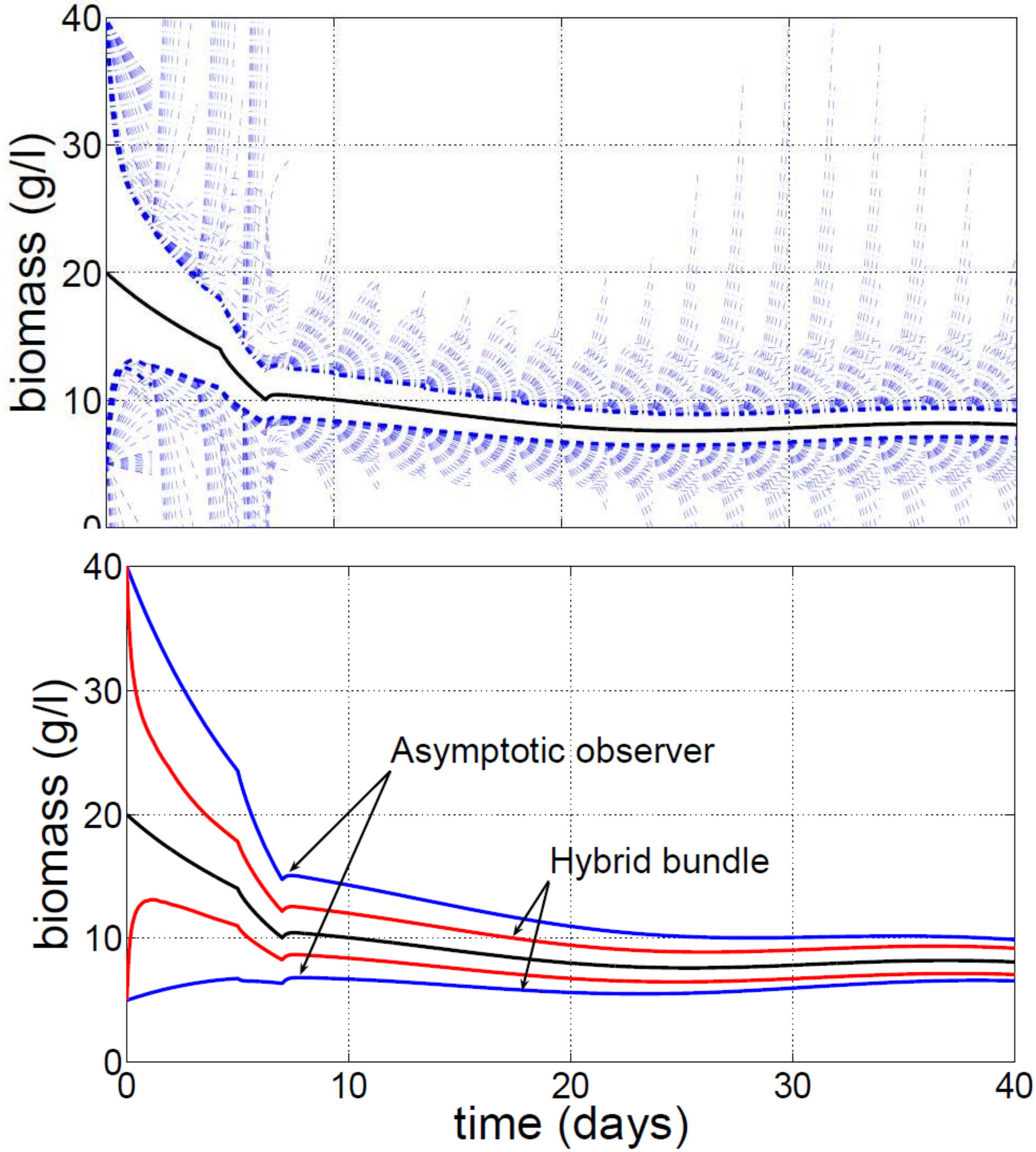
Optimal θ

Functional portrait:



Possibly optimal θ value

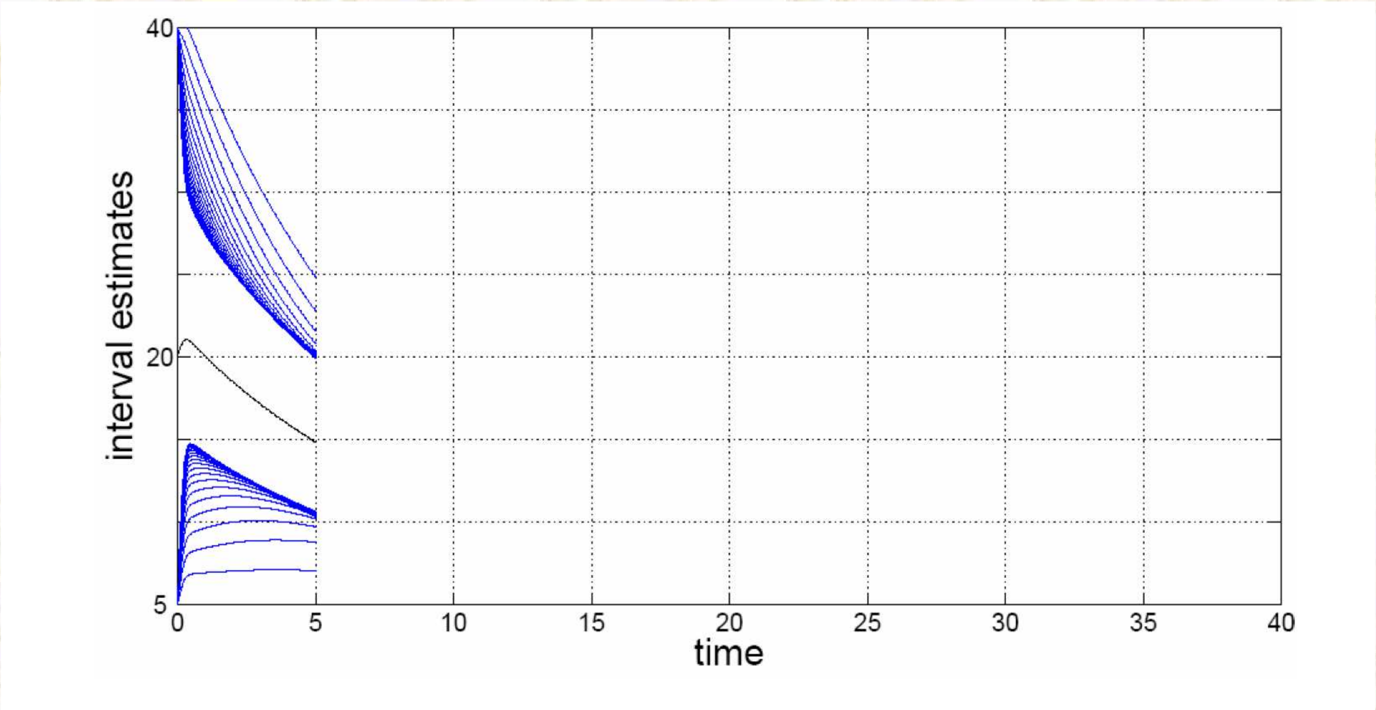
M. Moisan, O. Bernard, J.-L. Gouzé. Automatica, 2009



Best upper and lower estimates.

Amélioration: marche arrière

Direct time



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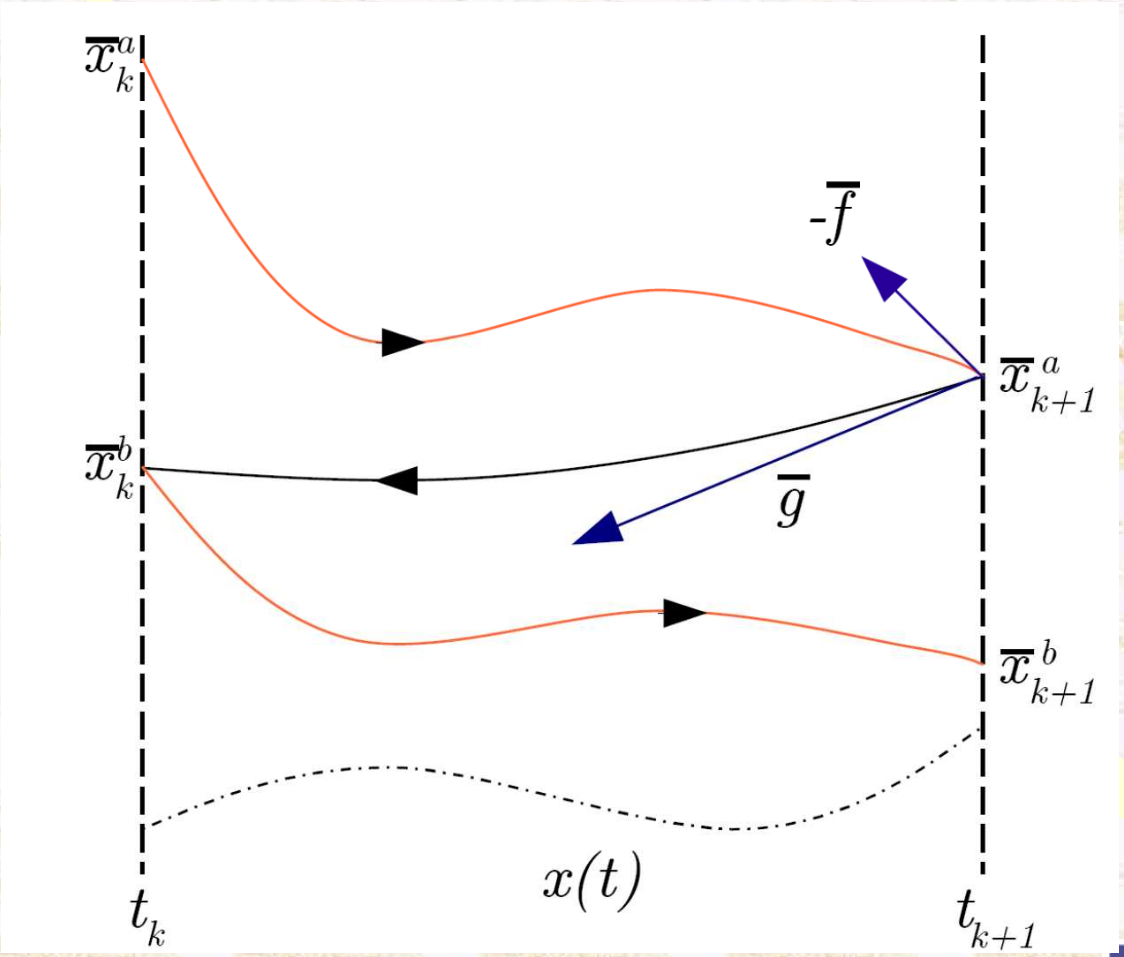
Reverse time



Reverse time observer

Consider $q_\gamma(t_r)$ the reverse time of variable $z(t)$, where $t_r = 2t_{k+1} - t$

$$\acute{q}_\gamma = (\gamma - 1)\mu(s)(q - \gamma s) + u(q - \gamma s_{in}) + \acute{\gamma} s, \quad {}^\prime = \frac{d}{dt_r}$$

Given $x_f = x(t_{k+1})$ such that $x_f \in [\underline{x}_f, \overline{x}_f]$ then

- For $\gamma < 0$

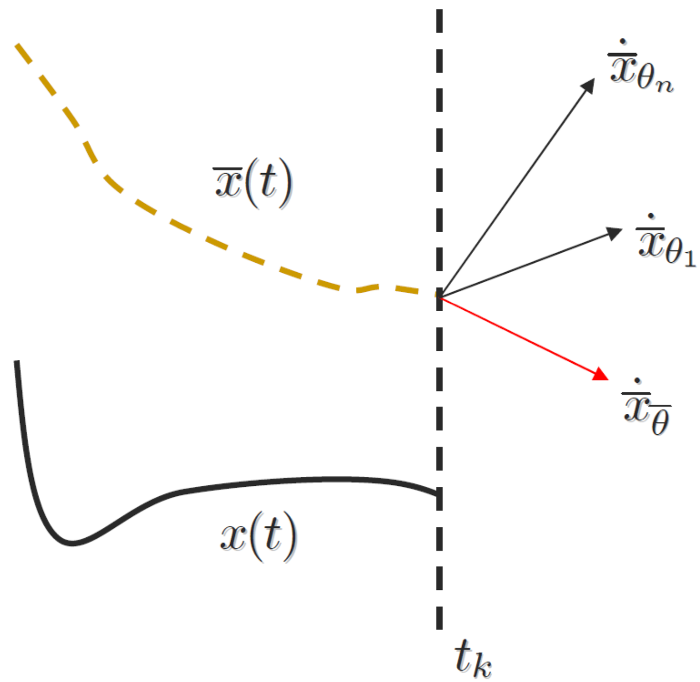
$$\begin{cases} \acute{\overline{q}}_\gamma = (\gamma - 1)(\underline{\mu}(s)\overline{q}_\gamma - \gamma \underline{\mu}(s)s) + u(\overline{q}_\gamma - \gamma s_{in}^+) + s\acute{\gamma} \\ \acute{\underline{q}}_\gamma = (\gamma - 1)(\overline{\mu}(s)\underline{q}_\gamma - \gamma \overline{\mu}(s)s) + u(\underline{q}_\gamma - \gamma s_{in}^-) + s\acute{\gamma} \end{cases}$$

observer 1

with $\underline{x}_\gamma(t_r) = (\underline{q}_\gamma(t_r) - \gamma s(t_r))/k$ and $\overline{x}_\gamma(t_r) = (\overline{q}_\gamma(t_r) - \gamma s(t_r))/k$

Reverse time

- There exists a 'best observer'.
- The optimal gain depends on the unknown state.
- The optimal gain is the solution of (upper bound) $\min_{\theta} \dot{\overline{x}}_{\theta}$

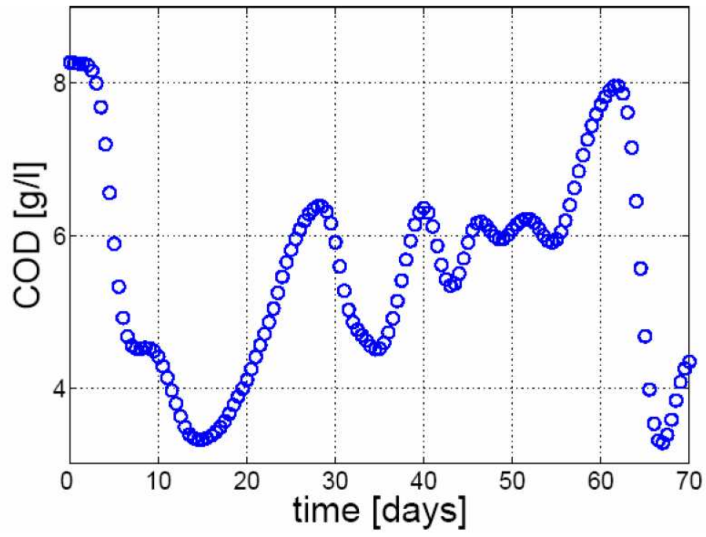


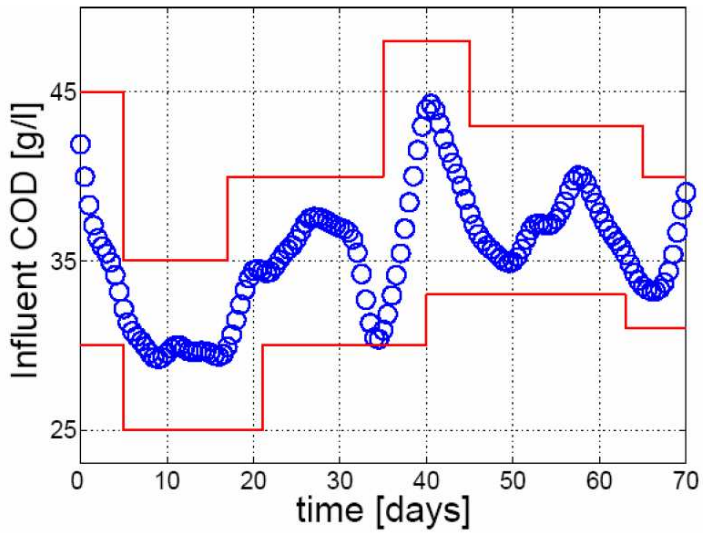
Direct time: The optimal gain belongs to an interval.

Reverse time: The optimal is a high and negative gain.

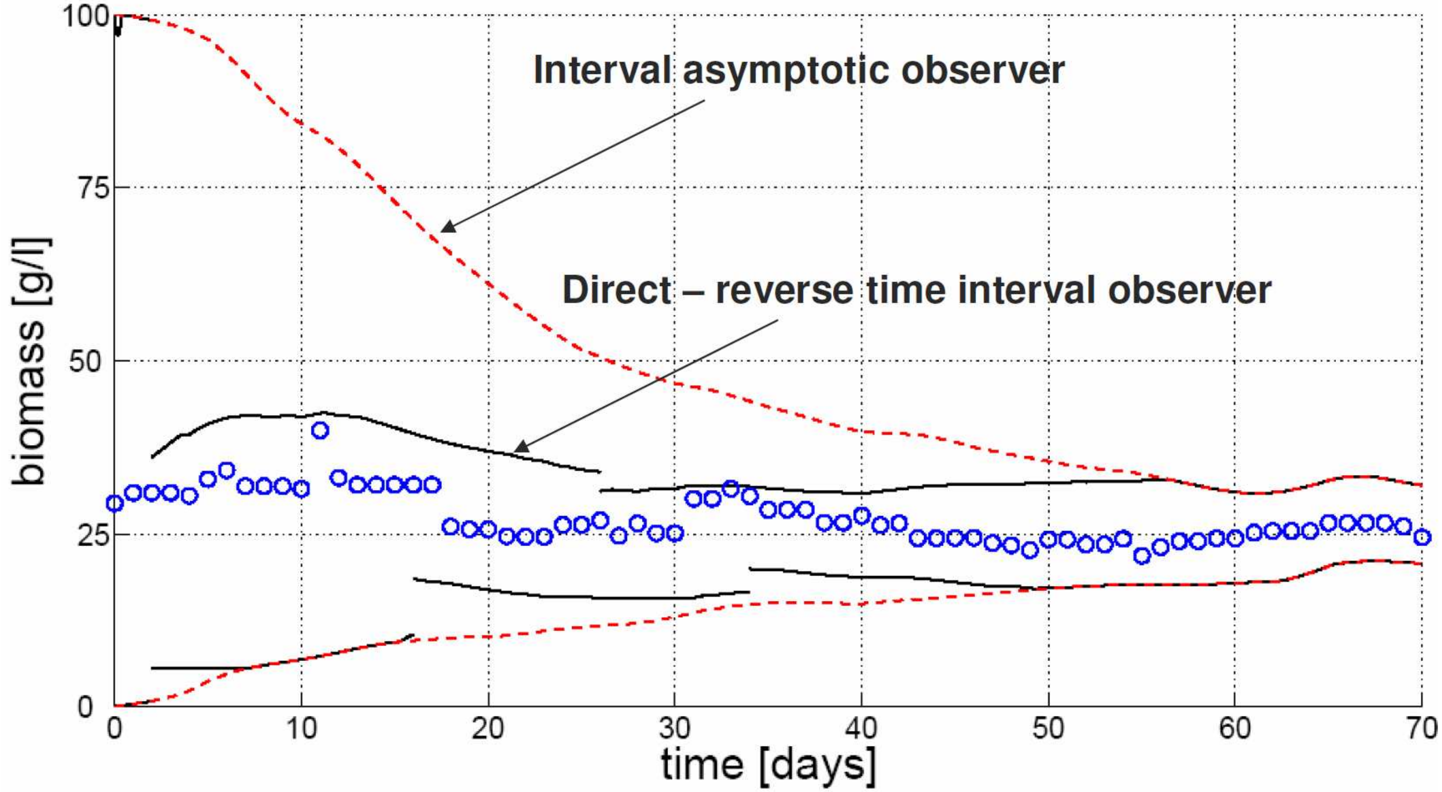
Application

- Anaerobic digestion: AGRALCO industrial fermenter (volume = 2000 m^3)
- Growth rate: Haldane function $\mu(s) = \dfrac{\mu_h s}{s + k_s + s^2/k_i}$
- Uncertainties: a) Influent substrate 30%, b) parameter $\mu_h \in [0.72, 1.08]$



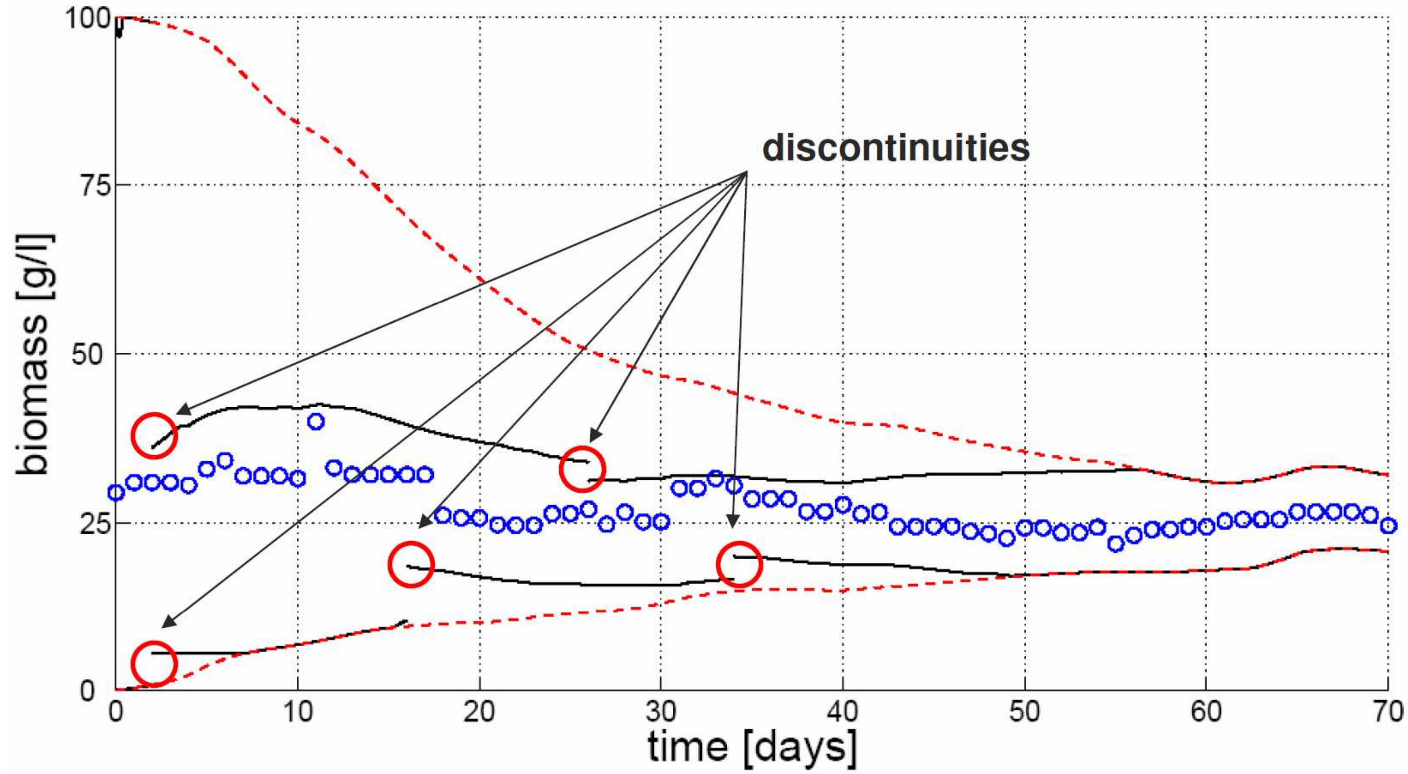


Industrial application



Interval estimates for the biomass

Industrial application



Interval estimates for the biomass

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When is the system cooperative after a change of variables ?

Two dimensional systems:

$$\begin{cases} \dot{\xi}_1 &= a_{1,1}\xi_1 + a_{1,2}\xi_2 + \phi_1(t)\ , \\ \dot{\xi}_2 &= a_{2,1}\xi_1 + a_{2,2}\xi_2 + \phi_2(t)\ , \end{cases}$$

where $\phi(t) = (\phi_1(t)\ \phi_2(t))^\top$ is a measurable function of time.

Dimension 2 - Assumptions

Basic assumptions:

• We will assume that the matrix $A = (a_{i,j})$ is Hurwitz.

• We will assume that there are ϕ^- and ϕ^+ such that, component by component,

$$\phi^-(t) \le \phi(t) \le \phi^+(t)$$

for all $t \ge 0$.

Consider the system (1) and assume that this system is cooperative. Then one can immediately propose an interval observers:

$$\begin{cases} \dot{z}_1^+ &= a_{1,1}z_1^+ + a_{1,2}z_2^+ + \phi_1^+(t) \ , \\ \dot{z}_2^+ &= a_{2,1}z_1^+ + a_{2,2}z_2^+ + \phi_2^+(t) \ , \\ \dot{z}_1^- &= a_{1,1}z_1^- + a_{1,2}z_2^- + \phi_1^-(t) \ , \\ \dot{z}_2^- &= a_{2,1}z_1^- + a_{2,2}z_2^- + \phi_2^-(t) \ , \\ x^+ &= z^+ \ , \ x^- = z^- \ . \end{cases}$$

Is the system cooperative after a basis change ?

There does not exist any (time invariant) change of coordinate if:

$$\mathrm{tr}(A)^2 - 2\,\det(A) \;\leq\; 0$$

BUT a time varying change of coordinate solves the problem !

F. Mazenc, O. Bernard, IEEE TAC, 2010,

Difficult case

- **Non real eigenvalues**

Canonical form: $$\begin{bmatrix} -\kappa & \omega \\ -\omega & -\kappa \end{bmatrix}$$

Rapidly rotating systems $\omega > \kappa$

Non cooperative system after any (linear) change of variable!

Time varying change of coordinates

Let $z = (z_1 \ z_2)^\top$ with $z = \lambda(t)\xi$, where

$$\lambda(t) = \begin{bmatrix} C(\omega t) - \aleph\frac{S(\omega t)}{\omega} & -\frac{a_{1,2}S(\omega t)}{\omega} \\ -\frac{a_{2,1}S(\omega t)}{\omega} & \aleph\frac{S(\omega t)}{\omega} + C(\omega t) \end{bmatrix} \quad (7)$$

with $\aleph = \frac{a_{11}-a_{2,2}}{2}$, $C(r) = \cos(r)$, $S(r) = \sin(r)$. Then for all t the matrix $\lambda(t)$ is invertible and in the z-coordinates, the system (1) becomes diagonal:

$$\dot{z} \;=\; -\kappa z + \psi(t)\ . \quad (8)$$

We have

$$\psi(t) = \lambda(t)\phi(t)$$

and, for all $t \geq 0$,

$$\psi^-(t) \leq \psi(t) \leq \psi^+(t)$$

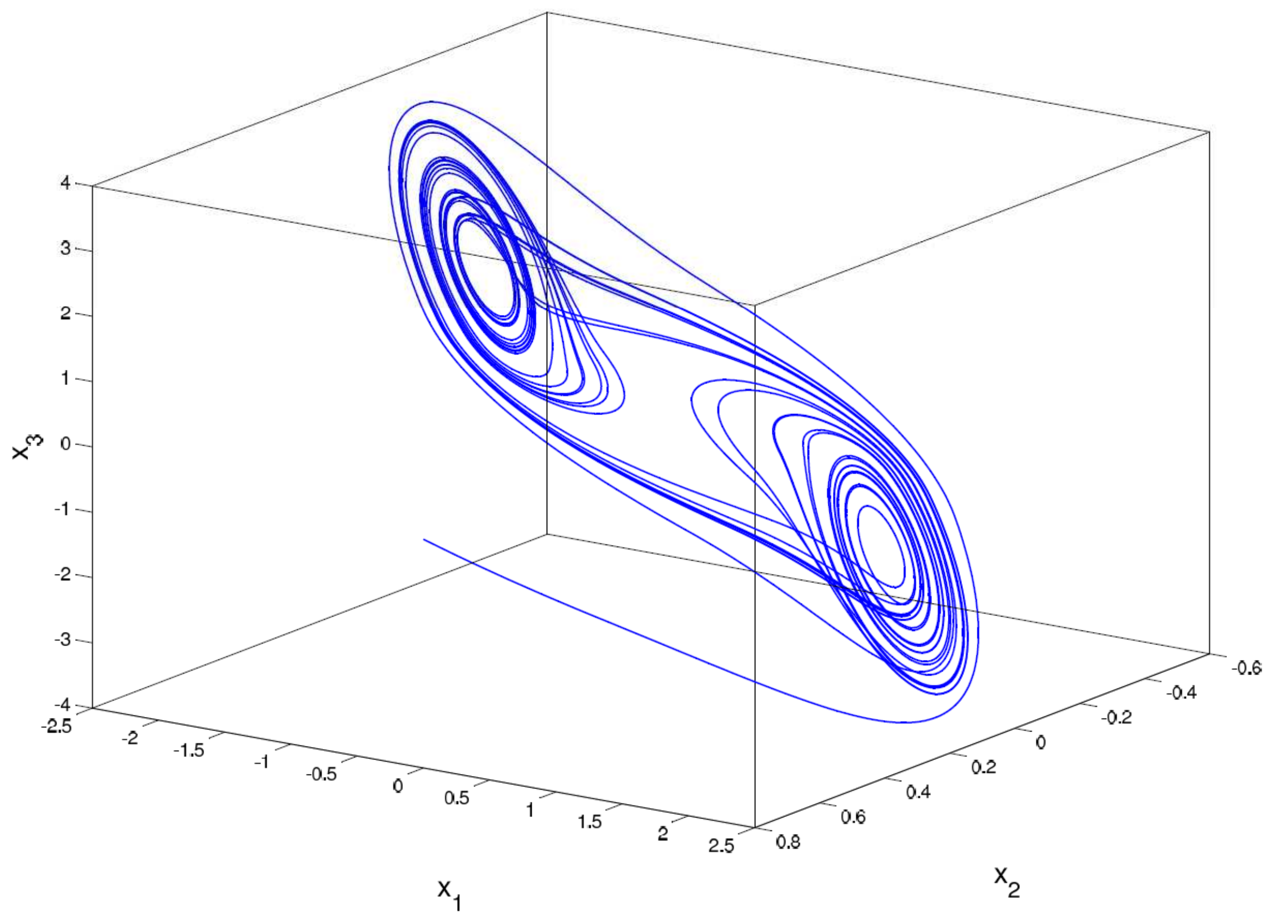
with

$$\psi^+(t) = \lambda_p(t)\phi^+(t) + \lambda_n(t)\phi^-(t) \ ,$$

$$\psi^-(t) = \lambda_n(t)\phi^+(t) + \lambda_p(t)\phi^-(t)$$

with $\lambda_p(t) = \max\{0, \lambda(t)\}$ and $\lambda_n(t) = \lambda(t) - \lambda_p(t)$.

Example: Chua's model



ems. Its

$(x_1)]$,

$|x_1 - 1|)$.

Chua's example

The system can be equivalently rewritten

$$\begin{cases} \dot{x}_2 &= -x_2 + x_3 + y \ , \\ \dot{x}_3 &= -\beta x_2 - \gamma x_3 \ , \\ \dot{y} &= \alpha x_2 - \alpha[y(1+b) + g(y)] \ . \end{cases}$$

Since y is measured, we focus on the (x_2, x_3)-subsystem (11), and define

$$\phi(t) = \begin{pmatrix} y(t) \\ 0 \end{pmatrix}, \; A = \begin{bmatrix} -1 & 1 \\ -\beta & -\gamma \end{bmatrix}.$$

The system cannot be transformed into a cooperative system through a time-invariant change of coordinates.

Major question: When (1) is not cooperative, what can be done ?

Classical answers:

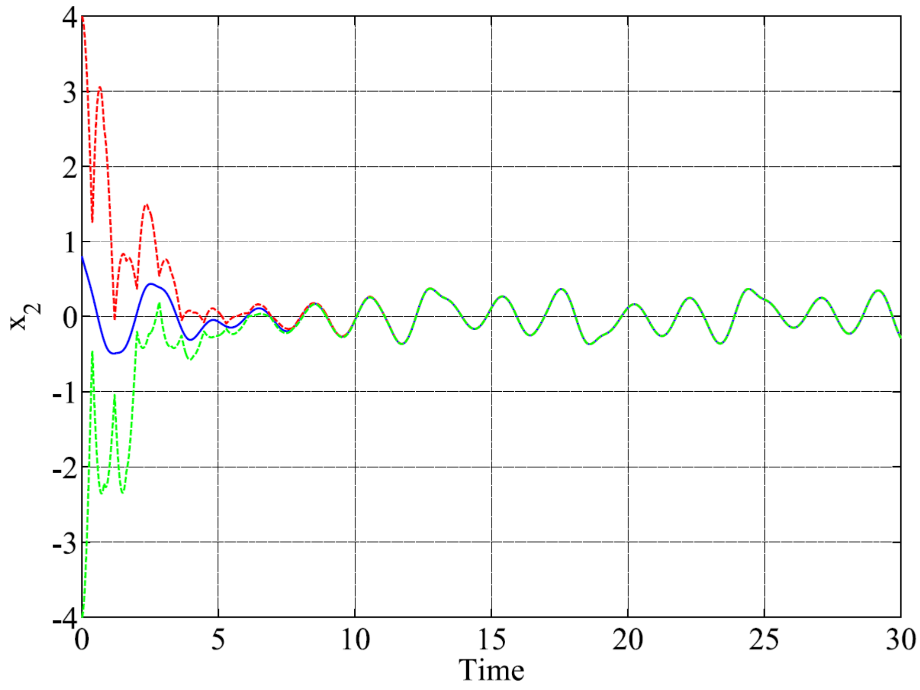
(i) framers can always be constructed *but in general they are not stable.*

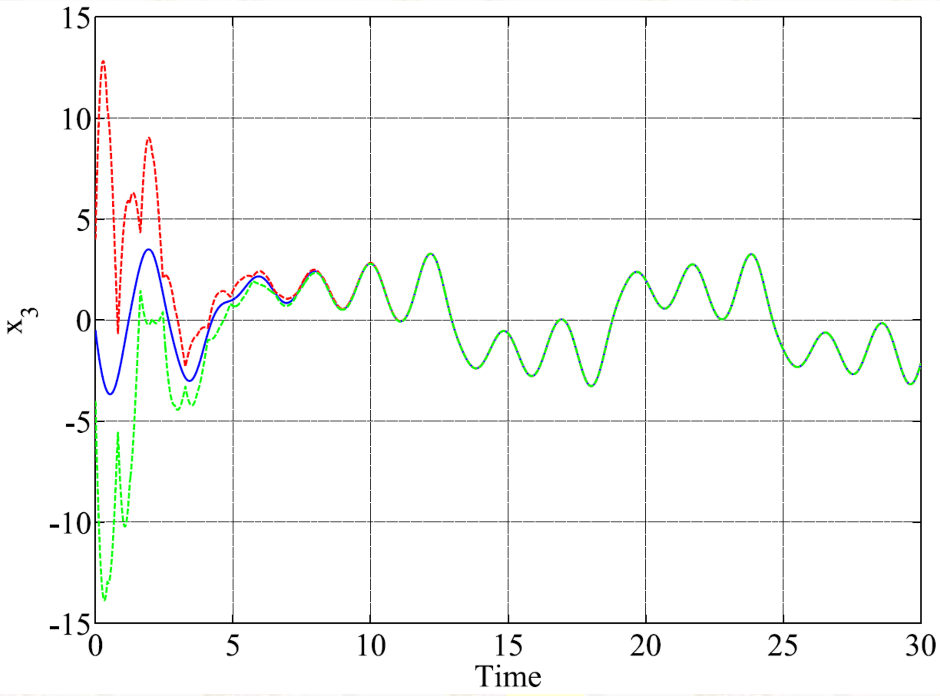
(ii) sometimes time-invariant changes of coordinates yield cooperative systems.

However, a time-varying periodic interval observer can be constructed thanks to the change of coordinates

$$
\begin{aligned}
z_1 &= \left(-\tfrac{-1+\gamma}{2}\tfrac{\sin(\omega t)}{\omega} + \cos(\omega t)\right) x_2 - \tfrac{\sin(\omega t)}{\omega} x_3 \ , \\
z_2 &= \beta \tfrac{\sin(\omega t)}{\omega} x_2 + \left(\tfrac{-1+\gamma}{2}\tfrac{\sin(\omega t)}{\omega} + \cos(\omega t)\right) x_3 \ .
\end{aligned}
$$

Simulation (no uncertainty)





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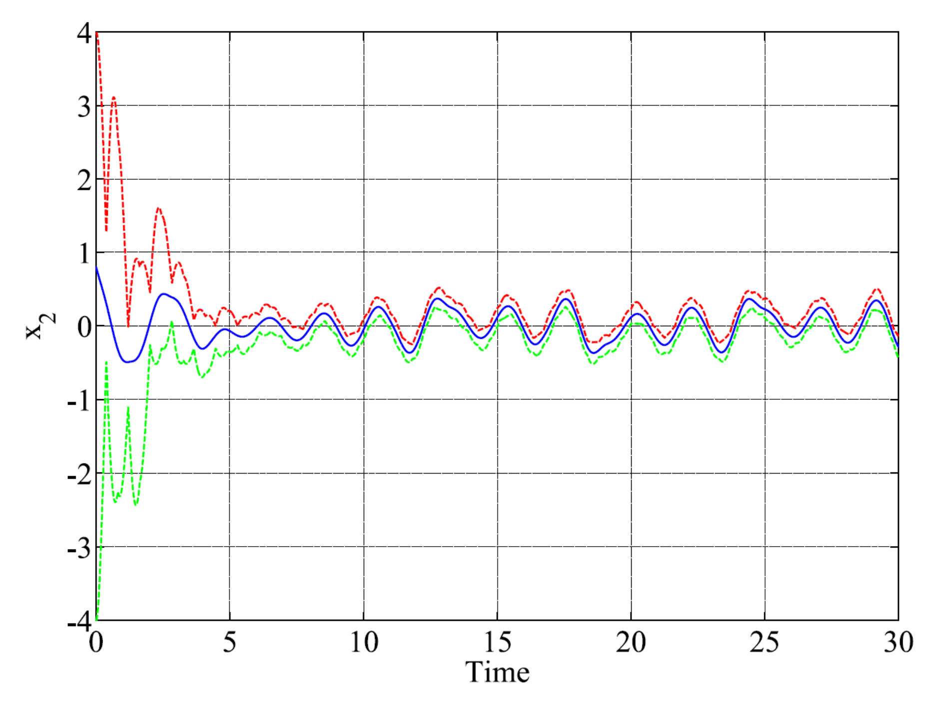
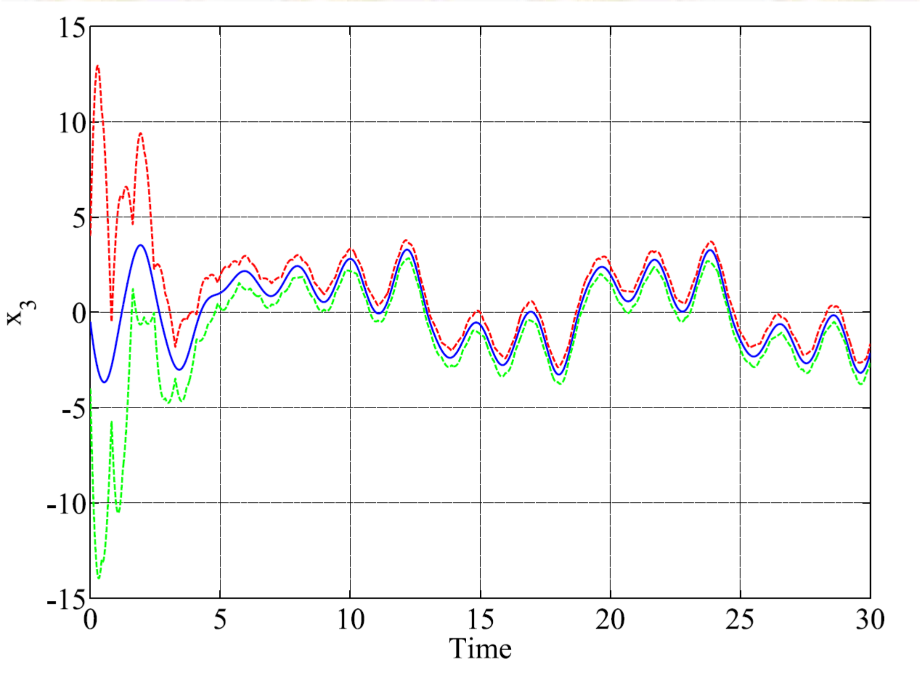
Simulation (uncertainties)



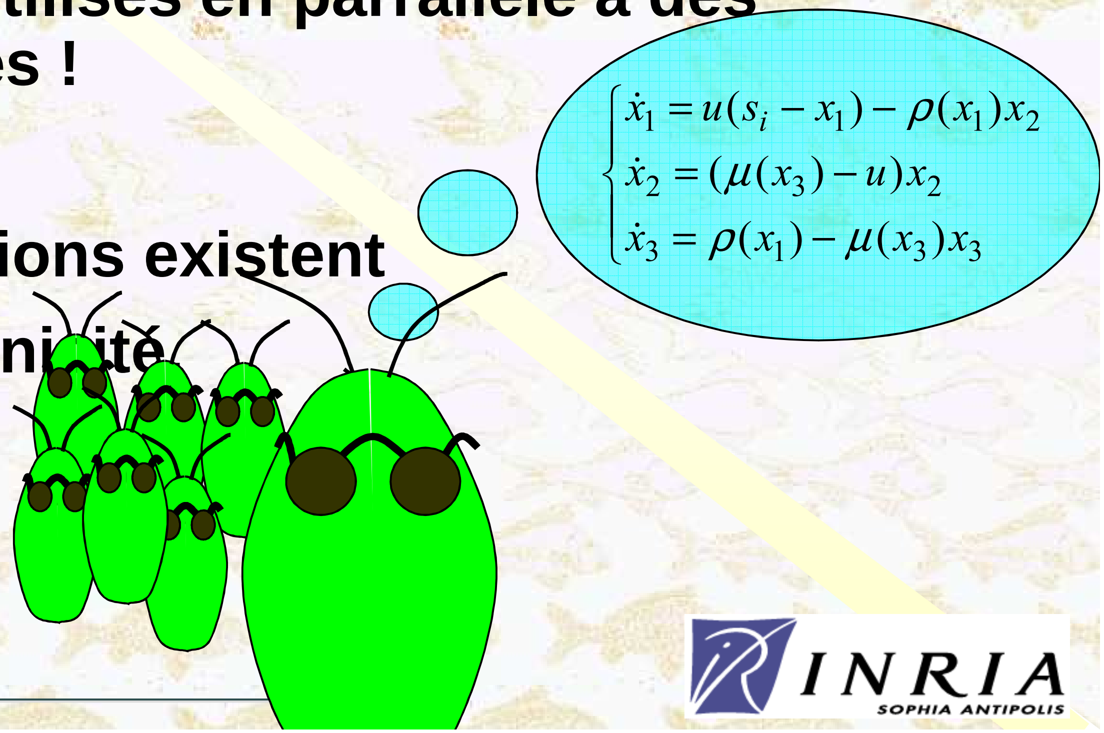


Conclusion

- **Les observateurs par intervalles permettent de gérer les incertitudes inhérentes aux systèmes biologiques**

- **Ils peuvent être comparés entre eux !**
- **Ils peuvent aussi être utilisés en parrallèle à des observateurs classiques !**

- **De nombreuses extensions existent (couplage, non monotonicité, linéaire dimension *n*, retard…)**



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