High gain observers with:
Linear, dynamics and homogeneous correction terms

V. Andrieu, *LAGEP-CNRS, université de Lyon*

Joint work with L. Praly (Mines de Paris) and A. Astolfi (Imperial College)

ENSMP, 2010

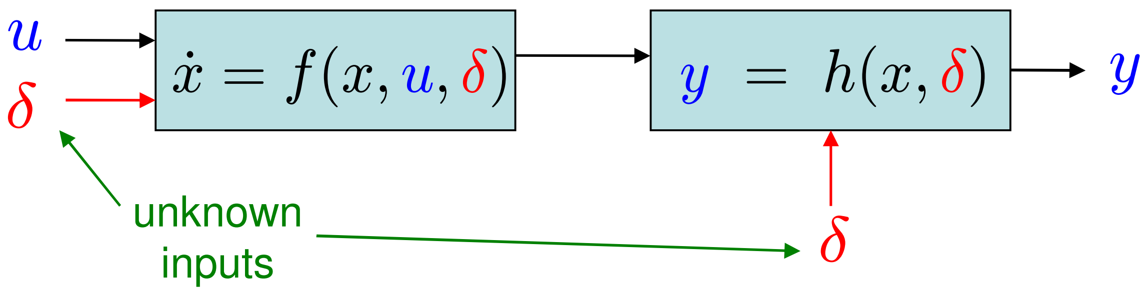
Consider a physical process with inputs and output :

ESTIMATION PROBLEM:

From the knowledge of past measurements and inputs $u(s), y(s) \quad s \in [0,t]$

give an estimate of the parameters describing the process

We suppose, we have a model of the physical process:



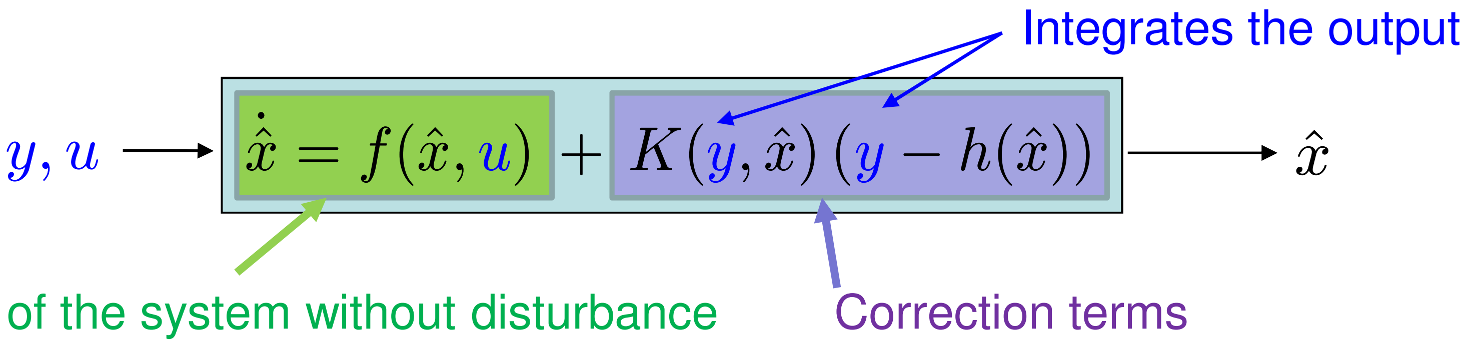
THE ESTIMATION PROBLEM BASED ON A MODEL :

From the knowledge of past measurements and known inputs $u(s), y(s)$, $s \in [0, t]$

find an algorithm which gives an estimate of the state $\hat{x}(t)$ *such that :*

1. $|\hat{x}(t) - x(t)|$ *remains "small" compare to* δ
2. $|\hat{x}(t) - x(t)|$ *goes to zero if* δ *goes to zero*

An *Kalman-filter-like-state-observer* can be a solution :



We want the correction term to be not too “big” and such that

$$\lim_{t\to+\infty} |\hat{x}(t) - x(t)| \;=\; 0$$

➡ Good robustness to measurement noise (if K is not too big)

Remarks : There are other type of observer which doesn’t take this form

1. Reduced order observers
2. Observers based on attractive and invariant manifold : Karagiannis-Astolfi, Kreisselmeir-Engel, Andrieu-Praly …
3. ...

Outline of the presentation

1. An illustrative example to get the main ideas
2. The main result
3. Simulations
4. Conclusion

Outline of the presentation

1. An illustrative example to get the main ideas
2. The main result
3. Simulations
4. Conclusion

Consider the system :

$$u \longrightarrow \boxed{\begin{array}{rcl}\dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & f_2(x_1, x_2, u)\end{array}} \longrightarrow \boxed{y = x_1}$$

with

$$f_2(x_1, x_2, u) = g(x_1)\, x_2 + x_2^{1+p} + u \quad , \qquad p \geq 0$$

Objective : We want K_1 and K_2 such that with the full order observer

$$y \longrightarrow \boxed{\begin{array}{rclcl}\dot{\hat{x}}_1 & = & \hat{x}_2 & + & K_1(y, \hat{x}) \\ \dot{\hat{x}}_2 & = & f_2(y, \hat{x}_2, u) & + & K_2(y, \hat{x})\end{array}} \longrightarrow \hat{x}$$

we have the property $(\hat{x}_1, \hat{x}_2) \to (x_1, x_2)$

Let's have a look to the dynamic of the error : $e = (e_1, e_2) = (\hat{x}_1 - x_1, \hat{x}_2 - x_2)$

Along the trajectories we have

$$\begin{array}{lcl} \dot{e}_1 & = & e_2 \\ \dot{e}_2 & = & \end{array} + \begin{array}{l} K_1(y,\hat{x}) \\ K_2(y,\hat{x}) \end{array} + \begin{array}{c} \\ f_2(y,\hat{x}_2,u) - f_2(y,\hat{x}_2 - e_2, u) \end{array}$$

Linear System

Correction terms

Nonlinear Part

HIGH-GAIN IDEA : ***Consider the nonlinear part as a disturbance !***

TWO STEPS :

1. Design the observer for the **linear part**
2. **Amplify the convergence** using high-gain techniques to increase robustness

We need an upper bound on this nonlinear disturbance :

$$f_2(y, \hat{x}_2, u) - f_2(y, \hat{x}_2 - e_2, u) = g(y)e_2 + \hat{x}_2^{1+p} - (\hat{x}_2 - e_2)^{1+p}$$

We investigate 3 cases :

CASE 1 : g is bounded, i.e. $g(y) \leq G$ and p=0

CASE 2 : only p=0

CASE 3 : no restriction

CASE 1 : If g is bounded, i.e. $g(y) \leq G$ and $p=0$, we get

$$|f_2(y, \hat{x}_2, u) - f_2(y, \hat{x}_2 - e_2, u)| \leq (G+1)|e_2|$$

LINEARLY bounded

⇒ We are in the globally Lipschitz case and we can use classical **LINEAR high-gain** design Introduced by Gauthier, Hammouri and Othman (IEEE-TAC 1992)

Classical linear high-gain design

Step 1 : design LINEAR correction terms for the linear part

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= \end{aligned} + \begin{aligned} k_1 e_1 \\ k_2 e_1 \end{aligned}$$

such that $(e_1, e_2) \to 0$

⇒ easy using linear tools !

Step 2 : Amplify convergence using an extra high-gain parameter :

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= \end{aligned} + \begin{aligned} L k_1 e_1 \\ L^2 k_2 e_1 \end{aligned} + f_2(y, \hat{x}_2, u) - f_2(y, \hat{x}_2 - e_2, u)$$

$$< |G+1||e_2|$$

If L big enough compare to the Lipschitz constant $G+1$, we get $(e_1, e_2) \to 0$

⇒ We have a high-gain observer

CASE 2 : If only p=0, we get

$$\boxed{\left|f_2(y,\hat{x}_2,u) - f_2(y,\hat{x}_2 - e_2,u)\right|} \leq \underbrace{(g(y)+1)|e_2|}_{\text{Output dependant but \textbf{LINEAR} incremental rate}}$$

⇨ The Lipschitz constant depends on the output.

Updated and linear high-gain design (Praly, IEEE-TAC 2003)

Step 1 : The same LINEAR correction terms

$$\begin{array}{rcl} \dot{e}_1 & = & e_2 \\ \dot{e}_2 & = & \end{array} + \begin{array}{c} k_1 e_1 \\ k_2 e_1 \end{array}$$

such that $(e_1, e_2) \to 0$

Step 2 : Amplify convergence accordingly to the Lipschitz constant

$$\begin{array}{rcl} \dot{e}_1 & = & e_2 \\ \dot{e}_2 & = & \end{array} + \begin{array}{c} L k_1 e_1 \\ L^2 k_2 e_1 \end{array} + f_2(y, \hat{x}_2, u) - f_2(y, \hat{x}_2 - e_2, u)$$

$$< |g(y) + 1||e_2|$$

L

$y \rightarrow$ $$\dot{L} = L(\varphi_1(\varphi_2 - L) + \varphi_3(g(y) + 1))$$

Output dependant incremental rate

with $L(0)>\varphi_2$, φ_2 and φ_3 big and φ_1 small

⇨ We have an observer **under the extra requirement that y is bounded**

About this gain adaptation

$$y \longrightarrow \boxed{\dot{L} = L(\varphi_1(\varphi_2 - L) + \varphi_3(g(y) + 1))}$$

If $L > \varphi_2 + \frac{\varphi_3}{\varphi_1}(g(y) + 1) \Longrightarrow \dot{L} < 0$

If $L < \varphi_2 + \frac{\varphi_3}{\varphi_1}(g(y) + 1) \Longrightarrow \dot{L} > 0$

$\Longrightarrow$ This says that L will "track" $\varphi_2 + \frac{\varphi_3}{\varphi_1}(g(y) + 1)$

L "follows" a linear combination of the Local incremental rate of the nonlinearity

CASE 3 : In the general case, we get

not linear anymore

$$|f_2(y,\hat{x}_2,u) - f_2(y,\hat{x}_2 - e_2,u)| \leq \underbrace{\left(|g(y)| + (1+p)|\hat{x}_2|^p\right)}_{\text{Local incremental rate}} |e_2| + \underbrace{|e_2|^{1+p}}_{\text{Power of the error}}.$$

⇒ We need robustness with respect to rational power of the estimation error.

⇒ **Polynomial correction terms** (=Homogeneous in the bi-limit framework)

⇒ The local incremental rate **depends on the output and the estimate.**

⇒ We **update** the high-gain parameter with the **output** and the **estimate**.

Updated and homogeneous high-gain design

STEP 1 : Design the correction term for the linear part

$$\begin{array}{rcl}\dot{e}_1 & = & e_2 \\ \dot{e}_2 & = & \end{array} + \begin{array}{c} K_1(e_1) \\ K_2(e_1) \end{array}$$

such that $(e_1, e_2) \to 0$

Employing Andrieu-Praly-Astolfi (SIAM 2008) we get the functions K

$$K_1(e_1) = -\left[\ell_1 e_1 + (\ell_1 e_1)^{\frac{1}{1-p}}\right]$$

$$K_2(e_1) = \ell_2 K_1(e_1) + \ell_2 K_1(e_1)^{1+p}$$

⇨ Gives **robustness** with respect to **polynomial** disturbance for a chain of integrator

STEP 2 : Amplify convergence accordingly to the Lipschitz constant

b is a new parameter

$$y \rightarrow \begin{array}{rcl} \dot{e}_1 & = & e_2 \\ \dot{e}_2 & = & \end{array} + \begin{array}{l} L^{1+b} K_1\left(\frac{e_1}{L^b}\right) \\ L^{2+b} K_2\left(\frac{e_1}{L^b}\right) \end{array} + f_2(y, \hat{x}_2, u) - f_2(y, \hat{x}_2 - e_2, u)$$

$L \uparrow \quad \downarrow \hat{x}_2$

$$< |g(y)| + (1+p)|\hat{x}_2|^p \, |e_2| + |e_2|^{1+p}$$

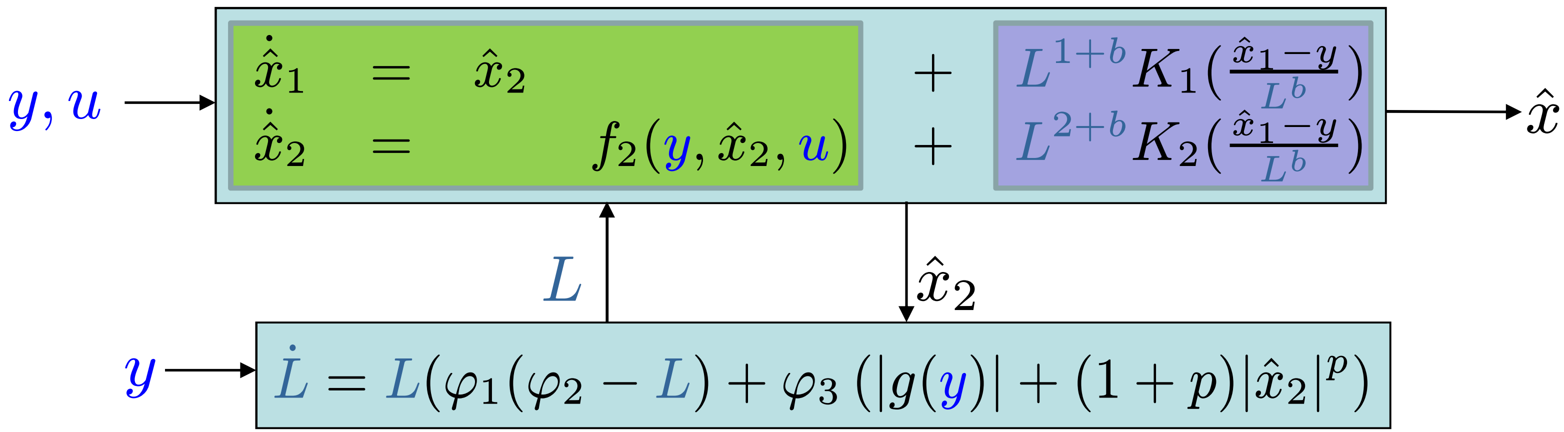
$$y \rightarrow \dot{L} = L(\varphi_1(\varphi_2 - L) + \varphi_3\left(|g(y)| + (1+p)|\hat{x}_2|^p\right)$$

Adaptation with the estimate

We have an observer under the extra requirement that y, and x_2 is bounded and p<1.

Restriction due to homogeneous in the bi-limit design

In conclusion the observer we obtain is



We get the property $(\hat{x}_1, \hat{x}_2) \to (x_1, x_2)$

Outline of the presentation

1. An illustrative example to get the main ideas
2. The main result
3. Simulations
4. Conclusion

Consider the system :

Following Gauthier and Kukpa, we can consider the case where we have $a_i(y)x_{i+1}$

Inputs: δ, u

$$\begin{aligned} \dot{x}_1 &= x_2 + f_1(u, y) + \delta_1(t) \\ &\vdots \\ \dot{x}_i &= x_{i+1} + f_i(u, y, x_2, \ldots, x_i) + \delta_i(t) \\ &\vdots \\ \dot{x}_n &= f_n(u, y, x_2, \ldots, x_n) + \delta_n(t) \end{aligned}$$

$$y = x_1 + \delta_y(t)$$

$\Rightarrow$ System in triangular form

Written in compact form :

$$\dot{x} = Sx + f(u, y, x) + \delta \longrightarrow y = x_1 + \delta_y(t)$$

The error system is :

Linear System **Nonlinear Part**

$$\dot{e} = Se + f(y, \hat{x}, u) - f(y, \hat{x} - e, u)$$

Note that if f is a C^1 function : $|f(a+b) - f(a)| \leq \Omega(a)\,|b| + \Delta(b)$

Local incremental rate — incremental rate for large b

Nonlinear bound Assumption : There exists d_∞ in $\left[0, \frac{1}{n-1}\right)$ and $v_j < \frac{1}{j-1}$ such that

$$|f_i(u,y,\hat{x}_2,\ldots,\hat{x}_i) - f_i(u,y,x_2,\ldots,x_i)|$$

$$\leq \Gamma(u,y)\left(1+\sum_{j=2}^{n}|\hat{x}_j|^{v_j}\right)\sum_{j=2}^{i}|\hat{x}_j - x_j|$$

$$+\, c_\infty \sum_{j=2}^{i}|\hat{x}_j - x_j|^{\frac{1-d_\infty(n-i-1)}{1-d_\infty(n-j)}}$$

polynomial bound on the local incremental rate

polynomial bound on the incremental rate for large error

⇨ The constraints $d_\infty < \frac{1}{n-1}$ and $v_j < \frac{1}{j-1}$ imposes a restriction on the degree of the polynomial

⇨ Comes from the use of homogeneity in the bi-limit

THEOREM : (Andrieu-Praly-Astolfi, Automatica 2006)

If the nonlinear bound is satisfied then we can construct K such the system

$$y, u \longrightarrow \quad \begin{array}{rcl} \dot{\hat{x}} & = & S\hat{x} + f(u, y, \hat{x}) + L\,\mathfrak{L}\,K\left(\frac{\hat{x}_1 - y}{L^b}\right) \\ \dot{L} & = & L\left[\varphi_1(\varphi_2 - L) + \varphi_3\,\Gamma(u, y)\left(1 + \sum_{j=2}^{n} |\hat{x}_j|^{v_j}\right)\right] \end{array} \quad \longrightarrow \hat{x}$$

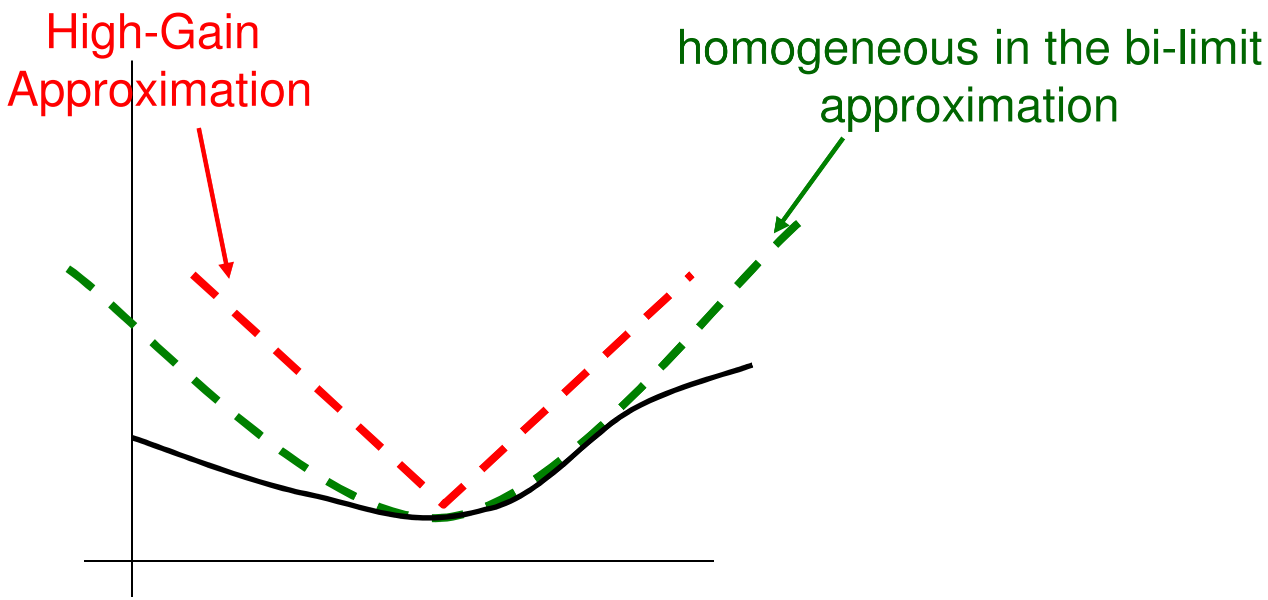
with $\mathfrak{L} = \mathrm{diag}(L^b, \ldots, L^{n+b-1})$ gives for **bounded solutions** the following property

1. If δ_y and δ are small we get $|\hat{x} - x|$ small
2. If δ_y and δ goes to 0, we get $|\hat{x} - x| \rightarrow 0$

About the result :

The result allows now to deal with **non globally Lipschitz systems**.

Even with globally Lipschitz system, this observer might give a **better performance** :



⇨ The bound fit the non linearity better then with the linear high gain approach

⇨ In principle, this means that we can use smaller correction term

⇨ **We expect better robustness to measurement noise**

Another result :

A result concerning high-gain observer for non linear system with bounded trajectories is already available (Lei-Wei-Lin, (2007))

With this approach no polynomial bound are required !

Ideas of this approach :

1 **linear** correction terms

2 **high-gain parameter strictly increasing** until we get convergence

⇨ No robustness property

In practice it might be difficult to implement

Outline of the presentation

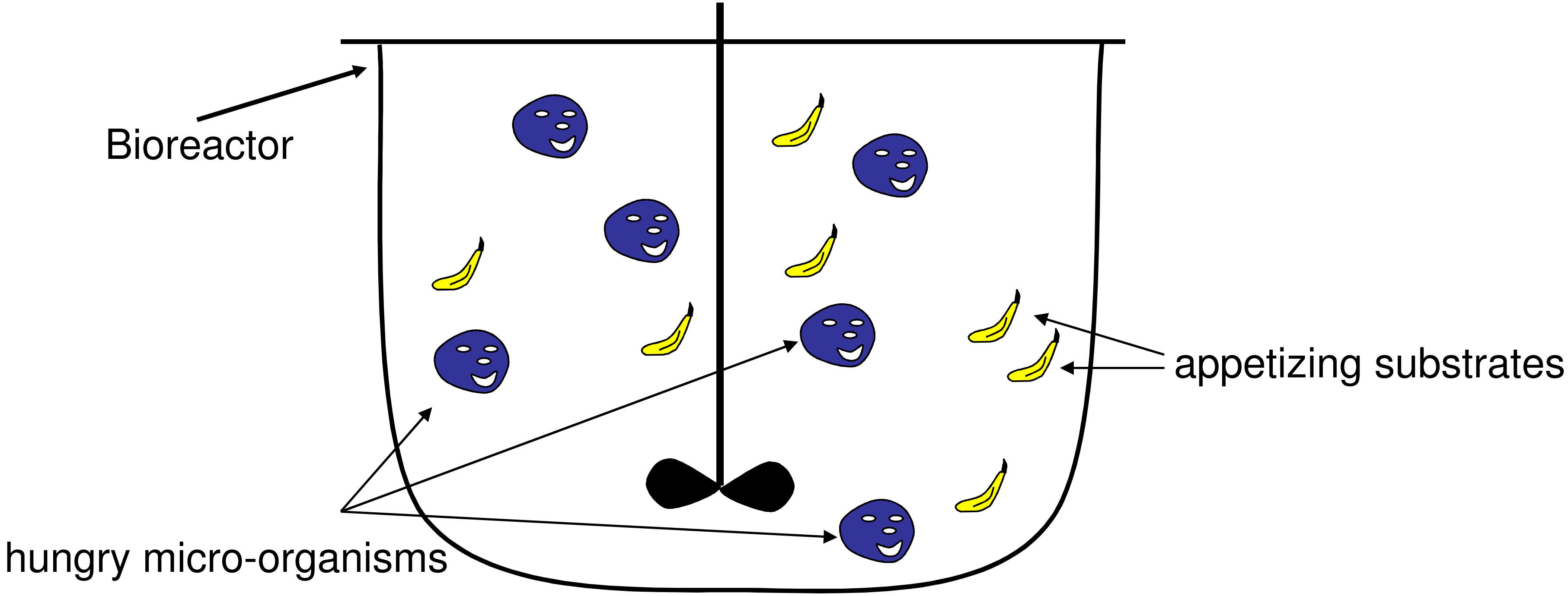
1. An illustrative example to get the main ideas

2. The main result

3. Application to a bioreactor

4. Conclusion

As an example, we consider a bioreactor



We measure the concentration of micro-organism.

⇒ What is the concentration of substrates in the bioreactor ?

As Gauthier Hammouri and Hotman (TAC-92), we consider the Contois model :

micro-organism concentration

substrates concentration

$$\begin{aligned}\dot{\eta}_1 &= \frac{\eta_1\eta_2}{k\eta_1+\eta_2} - u\eta_1\\ \dot{\eta}_2 &= -\frac{\eta_1\eta_2}{k\eta_1+\eta_2} + u(1-\eta_2)\end{aligned} \longrightarrow \boxed{y = \eta_1}$$

with $u \in M_u = [u_{\min}, u_{\max}] \subset (0,1) \quad , \qquad k \geq 0$

Remarks : Under the change of coordinate :$(\eta_1, \eta_2) \mapsto (x_1, x_2) = \left(\eta_1, \frac{\eta_1\eta_2}{k\eta_1+\eta_2}\right)$

The system is :

$$\begin{aligned}\dot{x}_1 &= x_2 - u\,x_1\\ \dot{x}_2 &= f_2(x_1, x_2, u)\end{aligned} \longrightarrow \boxed{y = x_1}$$

Triangular form

Remarks : There exists a forward invariant **compact** set :

$$M_\eta = \{(\eta_1, \eta_2) \;:\; \eta_1 \geq \epsilon_1,\; \eta_2 \geq \epsilon_2,\; \eta_1 + \eta_2 \leq 1\}$$

⇨ **We have a globally Lipschitz property.**

We can use the different approach :

1. Linear high-gain observer

$$|f_2(x_1, x_2, u) - f_2(x_1, \hat{x}_2, u)| \leq df_{2\,\max} \, |x_2 - \hat{x}_2|$$

2. Updated and linear high-gain observer

$$|f_2(x_1, x_2, u) - f_2(x_1, \hat{x}_2, u)| \leq \Omega_1(u, x_1, \hat{x}_2) \, |x_2 - \hat{x}_2|$$

3. Updated and homogeneous high-gain observer

$$|f_2(x_1, x_2, u) - f_2(x_1, \hat{x}_2, u)| \leq \Omega_2(u, x_1, \hat{x}_2) \, |x_2 - \hat{x}_2| + c_\infty |x_2 - \hat{x}_2|^{1+p}$$

Remarks :

$$\frac{\partial f_2}{\partial x_2}(x_1, x_2, u) \leq \Omega_2(u, x_1, x_2) \leq \Omega_1(u, x_1, x_2) \leq df_{2\,\max}$$

⇒ We expect better performance for the updated homogeneous high-gain observer

Parameter of the Simulation

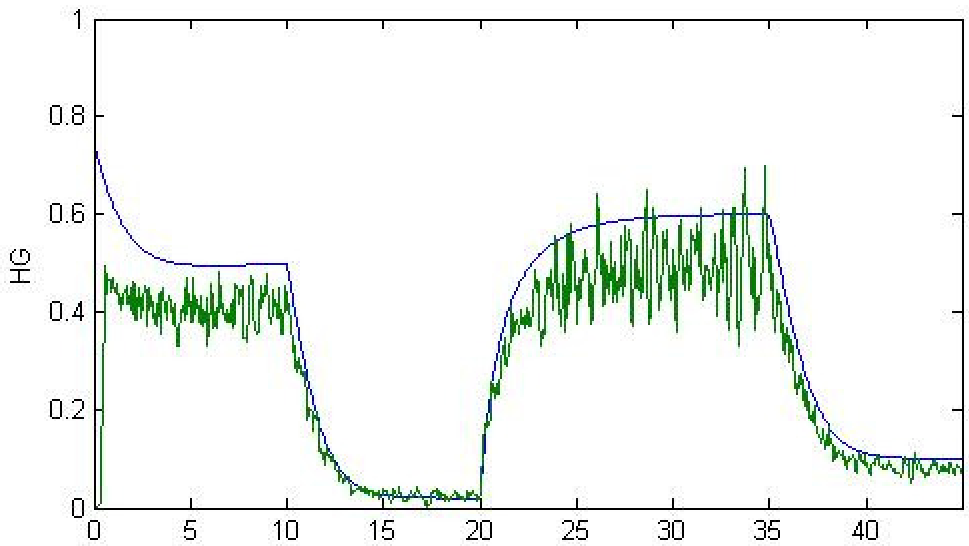
To evaluate the interest of this new observer, we consider **measurement noise** and **disturbance** in the simulation :

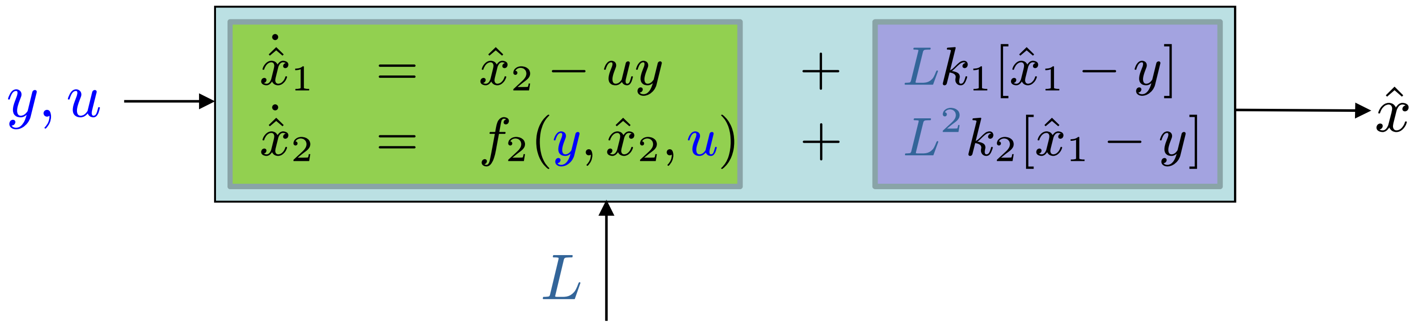
1. Parameter error : 20% error on the value of k
2. Gaussian white noise with standard deviation equal to 10% of the $\eta_1$domain

The control input is:

$$u(t) = 0.410 \text{ if } t < 10 , \quad = 0.02 \text{ if } 10 \leq t < 20 ,$$
$$= 0.6 \text{ if } 20 \leq t < 35 , \quad = 0.1 \text{ if } 35 \leq t .$$

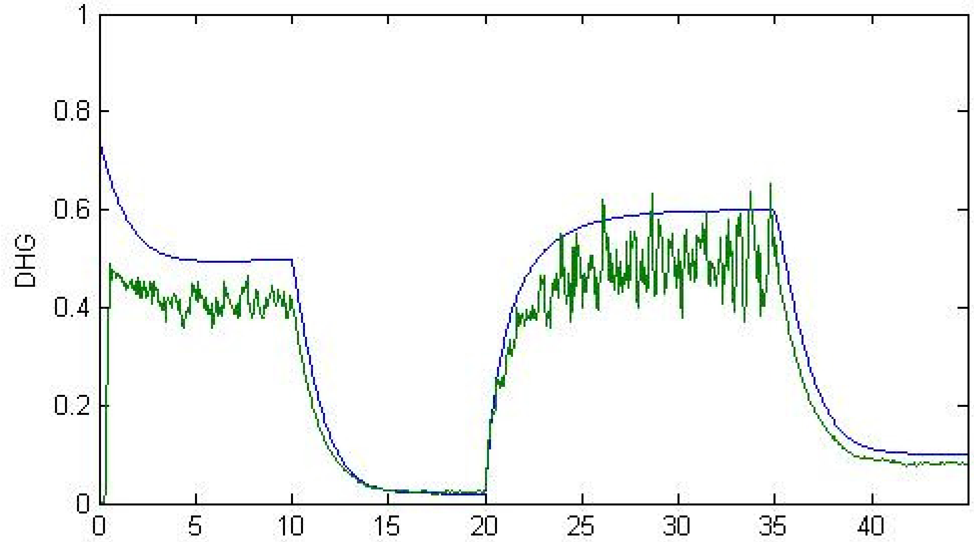
Result obtained with the **linear high-gain observer**

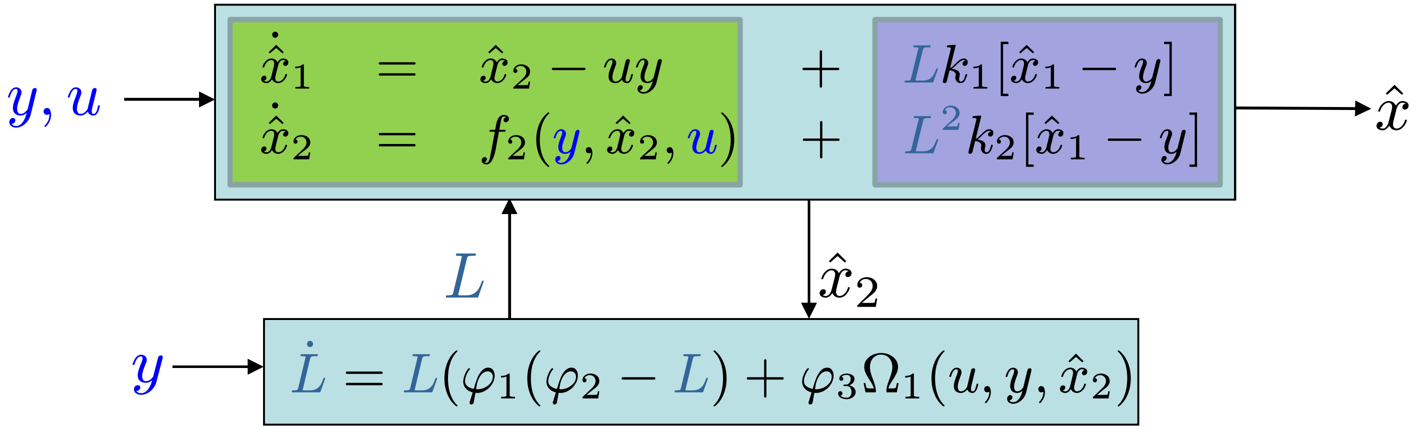




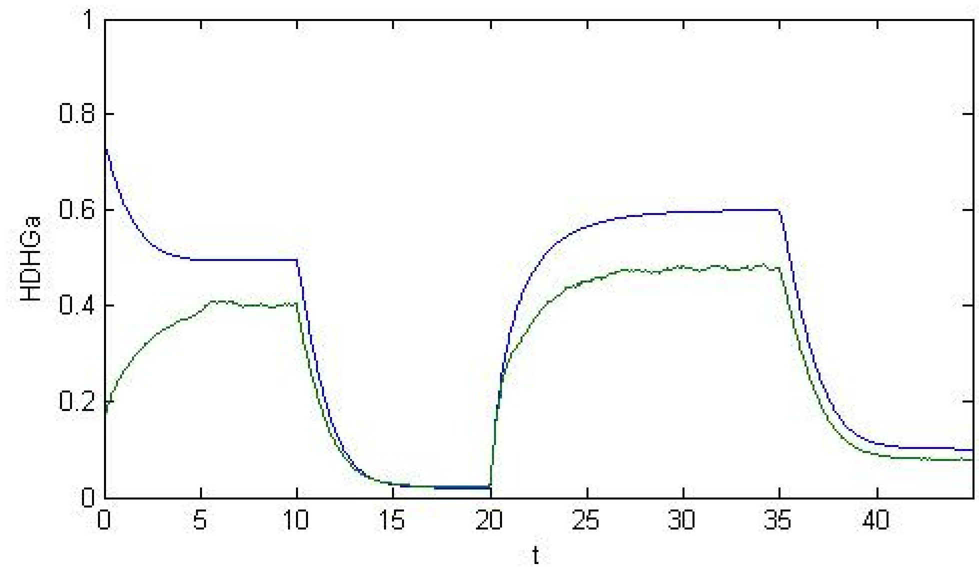
$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 - uy &&+ Lk_1[\hat{x}_1 - y] \\ \dot{\hat{x}}_2 &= f_2(y, \hat{x}_2, u) &&+ L^2 k_2[\hat{x}_1 - y] \end{aligned}$$

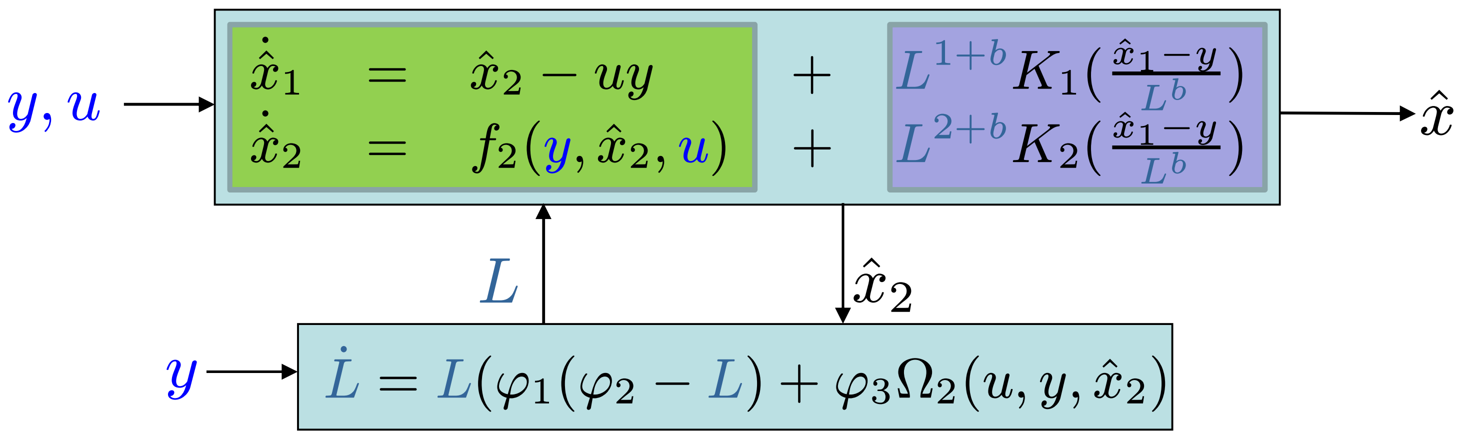
Result obtained with the **linear updated high-gain observer**





Result obtained with the **homogeneous updated high-gain observer**





Outline of the presentation

1. An illustrative example to get the main ideas

2. The main result

3. Application to a bioreactor

4. Conclusion

We have presented an extension of the classical high-gain observer

Novelties :

1. Allow **non globally Lipschitz nonlinearities**
2. Give **better performances** in terms of robustness

Techniques used :

1. Gauthier, Hammouri and Othman's high-gain methodology
2. Praly's gain adaptation
3. Homogeneous in the bi-limit tools

Our references on this subject :

1. Homogeneous in the bi-limit theoretical foundations published in SIAM 2009
2. This work is publised in Automatica 2009

THANK YOU !!!